EQUIVARIANT ALGEBRAIC GEOMETRY

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Note: this was a guest lecture by Arnav Tripathy.

1. Grothendieck-Riemann-Roch

Let $f\colon X \to Y$ be a smooth proper morphism of smooth schemes (these hypotheses are not optimal).

$$\begin{tikzcd} K_0(X) \arrow[r, "\mathrm{ch}\cdot \mathrm{Td}(X)"] \arrow[d, "\pi_!"'] & CH^*_{\mathbb{Q}}(X) \arrow[d, "\pi_*"'] \\ K_0(Y) \arrow[r, "\mathrm{ch}\cdot \mathrm{Td}(Y)"] & CH^*_{\mathbb{Q}}(Y) \end{tikzcd}$$

We want to give some example applications.

Example 1.1. Let $\mathcal{M}_g$ be the moduli space (stack) of genus g curves. As we saw last time, whenever you have a moduli space it's useful to understand its cohomology and Chow rings, as that gives you "characteristic classes." So we would like to understand the cohomology, Chow ring, etc. of $\mathcal{M}_g$ which will give a functorial attachment of invariants to a family of curves.

So how can we get a handle on $H^*(\mathcal{M}_g)$? Well, one explicit way is to study subvarieties on $\mathcal{M}_g$ (e.g. the locus of hyperelliptic curves), and this should be "Poincaré dual" to a cohomology class.

Another way is to use our knowledge of the moduli stack of vector bundles, and pull back classes from BGL_n. For instance, $\mathcal{M}_g$ has a universal curve $\mathcal{C}_g \xrightarrow{\pi} \mathcal{M}_g$, whose fiber over a point is precisely the corresponding curve. Let ω be the relative dualizing sheaf of $\mathcal{C}_g/\mathcal{M}_g$, i.e. and $\Omega = \pi_*\omega$. This has rank g, as the space of holomorphic differentials on a genus g curve is g-dimensional. Therefore, we obtain characteristic classes $\lambda_1, \dots, \lambda_g \in H^*(\mathcal{M}_g)$ where $\lambda_i := \mathrm{ch}_i(\Omega)$.

On the other hand, we have cohomology classes $c_i(\omega) \in H^*(\mathcal{C}_g)$ and then $\pi_*(c_i(\omega)) \in H^*(\mathcal{M}_g)$. Grothendieck-Riemann-Roch says that these two sets of cohomology classes should be related.

This is a prototypical example of GRR is used. You want to study cohomology classes on $\mathcal{M}_g$, and you obtain two collections via characteristic classes on some family, and you want to compare them. This is the GRR theorem.

Let's start the computation explicitly [see Harris-Mumford Moduli of curves]. According to GRR, $\mathrm{ch}(R\pi_1\omega) = \mathrm{ch}((\mathrm{Td}\,\omega^*)\,\mathrm{ch}(\omega))$. Now,

$$\mathrm{Td}\,\omega = \left(\frac{-\gamma}{1-e^{\gamma}}e^{\gamma}\right).$$

Well, $R^0\pi_*\omega = \Omega$, but what is $R^1\pi_*\omega$? It is $\mathcal{O}_{\mathcal{M}_g}$ - it is a family version of Serre duality, as over a point it just reflects Serre duality.

[For fun, we'd like to see this result from Grothendieck duality:

$$Rf_* = R\,\mathrm{Hom}(\mathcal{F}, f^!\mathcal{G}) \cong R\,\mathrm{Hom}(Rf_*, f_!\mathcal{G}).$$

Here $f^!\mathcal{G} = f^*\mathcal{G} \otimes \omega[\dim X - \dim Y]$. Taking all the sheaves to be the structure sheaves, we get

$$R\pi_*\omega = R\,\mathrm{Hom}(R\pi_*\mathcal{O}, \mathcal{O}).$$

It's not 100% clear.]

So the left hand side in GRR is

$$\mathrm{ch}(\Omega - \mathcal{O}) = 1 + \lambda_1 + \frac{\lambda_1^2 - \lambda_2}{2} + \dots$$

This is basically what all applicaitons of GRR look like. For a variant, one can consider the DM compactification $\overline{\mathcal{M}_{g,n}}$. Then one gets "$\psi$-classes" on $\overline{\mathcal{M}_{g,n}}$, which are line bundles $\mathcal{L}_i$ obtained by stitching together the cotangent bundle over the ith point (i.e. $p \rightsquigarrow T^*_{p_i}C$, i.e. the pullback of the cotangent bundle via the ith section), and $\gamma_i = c_1(\mathcal{L}_i)$.

2. Hirzebruch Riemann-Roch for stacks

Let $\mathcal{X} = [X/G]$ be a quotient stack which is smooth, proper, and Deligne-Mumford. Let $\mathcal{E}$ be a vector bundle on $\mathcal{X}$. We would like to be able to compute the holomorphic Euler characteristic

$$\chi(\mathcal{X}, \mathcal{E}) = \int_{\mathcal{X}} \mathrm{ch}(\mathcal{E})\,\mathrm{Td}(\mathcal{X}).$$

However, this is not quite right - that is the main difference between shemes and stacks. One instead has

$$\chi(\mathcal{X}, \mathcal{E}_1) = \int_{\mathcal{X}} \mathrm{ch}(\mathcal{E})\,\mathrm{Td}(\mathcal{X})$$

where $\mathcal{E}_1$ is the following. If we consider $\mathcal{E} \in K_0(\mathcal{X}) = K_G(X)$. This latter is a module over $K_G(*) = R(G)$ (the representation ring). So there is an element $1 \in \operatorname{Spec} R(G)$ (corresponding to the augmentation ideal, generated by virtual representations of dimension 0), and $\mathcal{E}_1$ is the component of $\mathcal{E}$ supported at 1. (It turns out that $\mathcal{E}$ is always supported at a finite number of points.)

So the integral only recovers the Euler characteristic over the identity component. Why should we have expected this to be true?

(1) The Chern character can only possibly "see" the component at 1. Since the right hand side only knows $\mathcal{E}$ through the component at 1, then we can't expect to "see" more on the left hand side.
(2) There are various equivariant cohomology theories, and the naïve equivariant cohomology theory localizes at 1. Namely, the naïve construction ("Borel-Moore") is if G acts on X then

$$H^*_{BM,G}(X) := H^*(X \times EG/G).$$

You can define this for any kind of cohomology theory, and this is how we defined the equivariant Chow theory. We claim that whenever we do this kind of construction, we localize at 1. But since the chern character maps to the Chow theory, which *is* constructed in the naïve way, it also must localize at 1.

Theorem 2.1 (Atiyah-Segal completion theorem)**.** *The map*

$$K_G(*) \to K(BG)$$

is an isomorphism after completing at the augmentation ideal.

Remark 2.2. The left hand side is $\mathrm{Rep}(G)$. The right hand side is the limit of K-theory in EG/G, i.e. the inverse limit of K-theory of $K((V_i - 0)/G)$ of a sequence of representations $V_i \hookrightarrow V_{i+1} \hookrightarrow \ldots$.

Example 2.3. For $G = \mathbb{G}_m$, $K_{\mathbb{G}_m}(*) = \mathrm{Rep}(\mathbb{G}_m) = \mathbb{Z}[t, t^{-1}]$. Now $\mathbb{G}_m$ acts freely on $\mathbb{A}^{n+1} - 0$, so $B\mathbb{G}_m = \lim_n \mathbb{P}^n$, and

$$K(B\mathbb{G}_m) = \varprojlim K(\mathbb{P}^n).$$

Now $K(\mathbb{P}^n)$ is generated by line bundles (argument: twist up to get a map in from a line bundle). We claim that $K(\mathbb{P}^n)$ is generated by $1, t^{\pm 1}, t^{\pm 2}, \ldots$. Why? There is an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to \mathcal{O}(2)^{\binom{n+1}{2}} \to \ldots \to \mathcal{O}(n+1) \to 0.$$

This is essentially the Koszul complex: the maps $\mathcal{O} \to \mathcal{O}(1)^{n+1}$, are multiplication by x_j, etc. This relation shows that

$$K(\mathbb{P}^n) = \mathbb{Z}[t]/(t-1)^{n+1}$$

so the inverse limit is the completion of $\mathbb{Z}[t]$ at $t-1$. But that's the same as the completion of the representation ring at $t-1$.

Upshot: the integral formula only computes the "part at 1." You need to perform some trickery to add up the other contributions.