EQUIVARIANT ALGEBRAIC GEOMETRY

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1. GRR FOR QUOTIENT STACKS

Let G be an algebraic group (linear and affine) acting on X. The Fulton-Baum-MacPherson Riemann-Roch says that there exists a functorial map

$$G(X) \xrightarrow{\tau_X} CH(X)$$

which specializes to GRR in the case of smooth varieties.

Edidin-Graham extend this to stacks: there exists a map

$$G(G, X) = G(X/G) \xrightarrow{\tau} CH(G, X) = CH(X/G)$$

such that

$$\begin{tikzcd} G(G,X) \arrow[rr, "\tau_X"'] \arrow[dr] & & CH(G,X) \\ & \widehat{G(G,X)} \arrow[ur, "\cong"'] & \end{tikzcd}$$

where $\widehat{G(G,X)}$ is the completion at the augmentation ideal of virtual coherent sheaves/vector bundles of rank 0.

This means that the map τ_X "loses information." For instance, if $G = \mathbb{Z}/5$ (constant group), then we lose the information at the points other than the origin.

Now this map should specialize to the one from GRR when X/G is is a smooth scheme.

Example 1.1. If $G = \mathbb{G}_m = \mathbb{C}^\times$, then $R(G) = \mathbb{Z}[s, s^{-1}]$. The augmentation ideal is generated by $(s-1)$. So the augmentation ideal cuts out the identity. Let X be a point, so $[X/G] = BG$. Then upon tensoring with $\mathbb{Q}$, the situation looks like

$$\mathbb{Q}[s, s^{-1}] \to \mathbb{Q}[[t]].$$

This is not an isomorphism, as expected. But if we set $u = s - 1$ and complete at u, then we get a factorization through $\mathbb{Q}[[u]]$. However, this is a "complicated" isomorphism, as $s \mapsto e^t$, so $u \mapsto e^t - 1$.

1.1. **Facts about Artin stacks.**

Definition 1.2. A morphism $\mathcal{X} \to \mathcal{Y}$ is *strongly representable* if for any map from a scheme $N \to \mathcal{Y}$, the pullback

$$\begin{tikzcd} N \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ N \arrow[r] & \mathcal{Y} \end{tikzcd}$$

We say that the morphism is representable if the same holds with N an algebraic space instead of a scheme.

Every Deligne-Mumford stack (finite type over k) has a *finite* cover by a scheme. This is non-trivial (the definition of an atlas doesn't involve properness) - see for instance [Edidin-Kresch-Hassett-Vistoli]. This is a handry crutch to use, along with an atlas. For instance, what does it mean for a morphism of stacks to be proper? The standard way of extending a definition from a scheme is to say that a representative morphism $\mathcal{X} \to \mathcal{Y}$ has property (P) if for all scheme maps $N \to \mathcal{Y}$, the base-change $N \times_{\mathcal{Y}} \mathcal{X} \to N$ is proper. However, this only works for representable morphisms, and there are settings of interest where we don't have this, e.g. $\overline{\mathcal{M}_g} \to \mathrm{pt}$. For testing properness, you can instead use a proper cover by a scheme (which is automatically representable, as a map from a scheme).

Example 1.3. If G is a finite group, then $BG \to \mathrm{pt}$, and G has a finite cover by pt (the universal bundle!), so the map $BG \to \mathrm{pt}$ must have degree $\frac{1}{|G|}$.

We now restrict our attentiont to DM stacks, so we can talk about proper morphisms.

If $\mathcal{X} \to \mathcal{Y}$ is a proper morphism of stacks, then there exists a pushforward $\pi_*\colon CH_*\mathcal{X} \to CH_*\mathcal{Y}$. Here, we crucially need rational coefficients. In particular, if $\mathcal{X}$ is proper over k then we get $CH_0(\mathcal{X}) \xrightarrow{\pi_*} CH_0(\mathrm{pt})$, which is traditionally denoted by $\pi_*\beta = \int \beta$.

By the way, how do you define Chow groups of stacks? Let's talk about quotient stacks first. The motivation is that in topology, you take a contractible space EG with a free G-action, and you set $BG = EG/G$. Then to get X/G, you form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & EG \arrow[d] \\ X/G \arrow[r] & BG \end{tikzcd}$$

So we want an "almost contractible" space on which G acts freely. Take a representation V such that G acts freely except the origin. Then we define

$$A^*(X/G) = A^*(X \times^G V) \text{ if } * < d.$$

One then defines the the Chow groups of X/G to be the "limit" of these, but one has to check that this is well-defined.

We also have a notion of flat pullback: there is a pullback functor on quasicoherent sheaves, and we demand that it be flat. This is okay, since it can be checked locally on an atlas of schemes.

Coarse moduli spaces. The *coarse moduli space* of a DM stack is an algebraic space X admitting a morphism $\mathcal{X} \to X$ that is initial with respect to morphisms to algebraic spaces, wich induces a bijection of geometric points.

One "motivation" for Riemann-Roch is to compute $h^0(\mathcal{L})$. Riemann-Roch computes the "approximation" $\chi(\mathcal{L})$, but we saw that we needed rational coefficients, which is a little disturbing. Anyway, the statement is that iif $\mathcal{X}$ is a DM stack, then there exists a map

$$G_0(X) \xrightarrow{\tau_X} CH^*(X) \otimes \mathbb{Q}$$

covariant for proper *representable* morphisms, and factors through $\widehat{G_0(X)} \otimes \mathbb{Q} \to CH^*(X) \otimes \mathbb{Q}$.

There are two issues.

(1) What is the augmentation ideal defining the completion?
(2) We want to apply this to the map $\mathcal{X} \to \mathrm{pt}$, which is generally non-representable.