EQUIVARIANT ALGEBRAIC GEOMETRY

FEBRUARY 9, 2015

1. An application of GRR

Let X be smooth and $\pi\colon V \hookrightarrow X$ the inclusion of an irreducible subvariety. Then $\mathrm{ch}([\mathcal{O}_V]) \in \mathrm{CH}(X)$, and we claim that it will be $[V]$ plus lower dimension terms. Intuitively this *should* be true, as both represent the "fundamental class" of V in some sense. However, we know of no easy proof.

Proof. If V is smooth, then we have a regular embedding and we can directly apply Grothendieck-Riemann-Roch:

$$\begin{tikzcd} K(V) \arrow[r, "\mathrm{ch}\cdot\mathrm{Td}"'] \arrow[d, "\pi_!"'] & CH(V) \arrow[d, "\pi_*"'] \\ K(X) \arrow[r, "\mathrm{ch}\cdot\mathrm{Td}"'] & CH(X) \end{tikzcd}$$

Then GRR tells us that $\mathrm{ch}([\mathcal{O}_V]) = \pi_*(\mathrm{Td}(N))^{-1}[V]$. As the Todd class is invertible, of the form 1 plus higher codimension terms, we get precisely $\pi_*[V] + \ldots$.

If V is not smooth, then at least it has a dense open subset which is smooth. Suppose that the complement is Z. We have a sequence

$$\mathrm{CH}(Z) \to \mathrm{CH}(X) \to \mathrm{CH}(X \setminus Z) \to 0$$

and we know the result in $\mathrm{CH}(X \setminus Z)$, and the rest is lower-dimensional terms, so we are happy.

$\square$

Let X be smooth. Then $\tau\colon K(X) \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(T_X)} \mathrm{CH}(X)$ is an isomorphism (rationally). In the 216 notes, we defined $\mathrm{CH} = \mathrm{Gr}(G(X))$ (graded by the dimension of the support of the sheaf), and this in turn is $\cong \mathrm{Gr}(K(X))$ (by smoothness of X).

Proof. One has a group homomorphism $\mathrm{CH} \to \mathrm{Gr}(K(X))$ sending $[V] \mapsto [\mathcal{O}_V]$ (one has to check that this is well-defined, which essentially boils down to the statement for two points in $\mathbb{P}^1$ - everything else follows because it is a pullback of this). Remark: we really needed to pass to the associated graded to kill the lower order terms, as $[V+W]$ should go to $[\mathcal{O}_V]+[\mathcal{O}_W]-[\mathcal{O}_{V\cap W}]$. Then the composition $\mathsf{K}(X)\to \mathsf{CH}(X)\to \mathsf{Gr}(\mathsf{K}(X))\to \mathsf{K}(X)$ is the identity modulo lower order terms, so that handles this composition.

What about the other composition? If you start with a coherent sheaf $\mathcal{F}$ on X, with rank r, then it is "mostly a vector bundle" in the sense that there exists a dense open subset $U\subset X$ such that $\mathcal{F}|_U$ is a vector bundle over U. Why? If you are "lucky" then there is a map $\mathcal{O}_X^r\to\mathcal{F}$ inducing an isomorphism on the generic point, and then the kernel and cokernel are supported on things on higher codimension. In general, one does some fiddling.

Why is this what we want? In $\mathsf{GrK}(X)$, we know that the composition gives back $[\mathcal{O}_V]$, but the argument shows that these things generate $\mathsf{K}(X)$. □

2. GRR FOR STACKS

2.1. **Baum-Fulton-MacPherson.** There is a generalization of GRR, by the name of the Baum-Fulton-Macpherson theorem, which says that for all finite type schemes over k there exists a homomorphism $G(X)\xrightarrow{\tau_X} G(Y)$ which is covariant for proper morphisms $\pi\colon X\to Y$ and it agrees with $\mathrm{ch}\cdot\mathrm{Td}(T_X)$ if X is smooth.

It is the case that

$$\tau_X([\mathcal{O}_V]) = [V] + \text{lower order terms}$$

(generalizing the claim from the beginning of class) and τ_X is an isomorphism. Probably these properties actually uniquely characterize τ_X.

We're now going to try to generalize this in stack territory.

2.2. **Quotient stacks.** Let G act on X. Then we form the quotient stack $[X/G]$. We have a homomorphism $G([X/G])\to \mathsf{CH}([X/G])$, which may not be an isomorphism, but it factors through an isomorphism which we will shortly describe:

$$\begin{tikzcd} G([X/G]) \arrow[rr] \arrow[dr] & & \mathsf{CH}([X/G]) \\ & ? \arrow[ur, "\cong"'] & \end{tikzcd}$$

The ring $\mathsf{K}([X/G])$ has an *augmentation ideal*, i.e. the ideal of objects of rank 0. For example, $\mathsf{L}(BG)$ is the representation ring of G, and and $\mathsf{K}(BG)$ has the augmentation ideal of rank 0 virtual representations. This is the "universal case," as $X\to \operatorname{Spec} k$ is flat and so we get a flat map $[X/G] \to BG$, and the pullback of the augmentation ideal of $K(BG)$ is that for $K([X/G])$. The ? above is $\widehat{G}([X/G])$, the completion of $G([X/G])$ at the augmentation ideal.

Theorem 2.1 (Edidin-Graham)**.** *This is an isomorphism.*

$$\begin{tikzcd}
G([X/G]) \arrow[rr, "\mathrm{ch}\cdot\mathrm{Td}(T_x)/\,\mathrm{Td}(\mathfrak{g})"'] \arrow[dr] & & CH([X/G]) \\
& \widehat{G}([X/G]) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

This is reasonsable if you think of $T_{[X/G]} = T_x/\mathfrak{g}$.