# MATH 245 CLASS 6 (DAN EDIDIN)

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## 1. Explaining why $K_0 = G_0$

We now prove the statement given earlier.

**Theorem.** $K_0(X) = G_0(X)$ on smooth varieties.

Key point: Every coherent sheaf has a finite resolution by locally free sheaves.

*Proof.*

We have a map $K_0(X) \to G_0(X)$.

Easy direction: This is surjective. Reason: Any coherent has a finite resolution by vector bundles, so its class in $G_0(X)$ is in the image of $K_0(X)$.

Hard direction:

Define $G_0(X) \to K_0(X)$. Idea: $[\mathcal{F}] \to \chi(P_\bullet)$ (where by $\chi(P_\bullet)$ we mean the alternating sums of the terms of the projective resolution).

Two things need to be checked.

(A) the definition is independent of resolution, and
(B) that the definition is additive.

This is a purely "categorical" result now. The result is now: $\mathcal{A}$ is an abelian category (in our case, coherent sheaves), $\mathbf{P} \subset \mathbf{A}$ is an exact subcategory (in our case, vector bundles). They satisfy:

(1) If $P_1 \twoheadrightarrow P_0$ is a surjection of objects in $\mathbf{P}$, then the kernel is also in $\mathbf{P}$.
(2) Every $A \in \mathbf{A}$ has a finite resolution in $\mathbf{P}$.

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Then $\mathsf{K}_0(\mathbf{P}) = \mathsf{K}_0(\mathbf{A})$. (Quillen has a generalization that says that the two spectra are homeomorphic.)

(The only proof in the literature of the easier statement seems to be Borel-Serre (1958) (refererd to in Fulton's Intersection Theory App. B.8.3).)

Two key facts:

Suppose $\mathcal{F}' \to \mathcal{F}$ is a surjection.

If $\mathcal{P}_\bullet \to \mathcal{F}$ is a finite resolution (in $\mathbf{P}$), then there exists a finite resolution $\mathcal{P}'_\bullet \to \mathcal{F}''$, and epimorphism $\mathcal{P}'_\bullet \to \mathcal{P}_\bullet$.

If $\mathsf{P}_\bullet \mathcal{F}$ and $\mathcal{P}'_0 \to \mathcal{F}$ are 2 resolutions, then there exists a resolution $\mathcal{P}'' \to \mathcal{F}$ with $\mathcal{P}''_\bullet \to \mathcal{P}'_\bullet$ and $\mathcal{P}'' \to \mathcal{P}_\bullet$ both surjective

**Lemma. (less general form of Borel-Serre Lemma 14).**

Suppose we have $0 \to \mathcal{K} \to \mathcal{P} \to \mathcal{F} \to 0$ and $0 \to \mathcal{K}' \to \mathcal{P}' \to \mathcal{F} \to 0$, then we have a short exact sequence $0 \to \mathcal{K}'' \to \mathcal{P}'' \to \mathcal{F} \to 0$, which surjects onto each of them.

*Proof sketch of Lemma.*

We take the fibered product:

$$\begin{tikzcd} \mathcal{P} \times_{\mathcal{F}} \mathcal{P}' \arrow[r] \arrow[d] & \mathcal{P}' \arrow[d] \\ \mathcal{P} \arrow[r] & \mathcal{F} \end{tikzcd}$$

This is the kernel of $\mathcal{P} + \mathcal{P}' \to \mathcal{F}$.

Now we know that this thing has a surjection from an element of $\mathbf{P}$. Call this $\mathcal{P}_2$.

We will now choose $\mathcal{P}_3$ as an element of $\mathbf{P}$ surjecting to $\mathcal{K}$, and $\mathcal{P}'_3$ (similarly).

Then $\mathcal{P}'' = \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}'_3$, and define correct map $\mathcal{P}'' \to \mathcal{F}$.

*Remark:* In Weibel's K-theory, Weibel gives a seemingly simpler approach that Dan couldn't make to work.

Daniel Litt's explanation: You wan tto show that the Euler characteristics of two truncated resolutions are the same, in $\mathsf{P}$. This is the same as showing that the Euler characteristic of the cone is $0$. This is the same as showing that the Euler charactersitic of an exact sequence in $\mathbf{P}$ is $0$. But then I can break this exact sequence into short exact sequences (working right-to-left), where we use the fact about kernels.

To make this work, we need a map from one projective resolution.

Weibel does that for us.

Consider the equalizer exact sequence $\mathcal{F} \to^{\Delta} \mathcal{F} \oplus \mathcal{F} \to\to \mathcal{F}$ (I can't quickly put one rightarrow on top of another!). We have $\mathcal{P}_\bullet \oplus \mathcal{P}'_\bullet$ mapping to $\mathcal{F} \oplus \mathcal{F}$, and we pull it back to $\mathcal{F}$, and this maps to both $\mathcal{P}_\bullet$ and $\mathcal{P}'_\bullet$.

## 2. Back to equivariant K-theory

$G = T$ is a torus (for notational simplicity). We have equivariant Riemann-Roch:

$$G_0(G, X)_{\mathbb{C}} \xrightarrow{\sim} \bigoplus_{h \in T} G_0(G, X)_{\mathfrak{m}_h}$$

$$\text{with} \qquad G_0(G, X) \xrightarrow[\sim]{\tau_X} CH^*_G(X)_{\mathbb{C}}$$

Localization: $G_0(G, X)_{\mathfrak{m}_h} = G_0(G, X^h)_{\mathfrak{m}_h}$.

Why is this helpful? Answer: because of twisting.

Consider $G \to G$ given by $g \mapsto hg$. There is an induced action on the coordinate ring $k[G] = R(G)_{\mathbb{C}} = K_0(G, pt)$.

Defines $t^\sharp : R(G)_{\mathbb{C}} \to R(G)_{\mathbb{C}}$, which takes $\mathfrak{m}_h \to \mathfrak{m}_1$.

When $h$ acts trivially, then we can lift this automorphism to K-theory.

If $Y$ is a $G$-space, and $h$ acts trivially on $Y$, and $h$ has finite order.

We lift $t^\sharp_h$ to an aautomorphism of $G_0(G, Y)$

Lift $t^\sharp_h$ to an automorphism of $G_0(G, Y)$. Then if $\mathcal{F}$ is a $G$-coherent sheaf, and $h$ acts trivially, then $\mathcal{F}$ breaks into a direct sum of eigenspaces of the finite cyclic group $\langle h \rangle$.

$t^\sharp_h([\mathcal{F}]) = \oplus_\chi \chi(h) f_\chi$.

E.g. twist by $e^{\pi i/2}$.

$1 + \chi \mapsto 1 + e^{i\pi/2}\chi$

$1 + \chi^2 \mapsto 1 - \chi^2$.

etc.

Thus if $h$ acts trivially, then the rank is the same at $h$ as at $1$.

Then we have an isomorphism

$$G_0(G, X)_{\mathfrak{m}_h} \xrightarrow{\sim} G_0(G, X^h)_{\mathfrak{m}_h} \xrightarrow{t^\sharp_h} G_0(G, X^h) \xrightarrow{RR} CH^*_G(X^h)_{\mathbb{C}}$$

so it gives an isomorphism

$$G_0(G,X) = \oplus G_0(G,X)_{\mathfrak{m}_h} \to \oplus_h CH^*_G(X^h) = CH^*_G(I_G X)$$

Now suppose G acts on X (acting tamely), such that $\mathcal{X} := [X/G]$ is a proper stack. The quotient $X/G$ is a proper algebraic space.

$\chi : G_0(G,X) := G_0(\mathcal{X}) \to G_0(X/G) \to G_0(pt) = \mathbb{Z}$.

$\chi(\mathcal{F}) = \sum(-1)^i h^i(X,\mathcal{F})^G$, where $\mathcal{F} \in G_0(G,X)$.

This is in fact $\chi(\pi_* f^G)$ where $\pi; X \to X/G$ is the quotient map.

Suppose X is smooth. We want to compute $\chi(\mathcal{F})$ using some intersection theory.

For example, say $\mathcal{X} = \mathbb{P}(1,2)$, where $\mathbb{A}^2 - \{0\}/\mathbb{C}^*$ (with the torus acting with weights $(1,2)$).

At this point, we change $\chi$ to $\xi$ to avoid too many $\chi$'s.

Now $K_0(\mathcal{X}) = \mathbb{Z}[\xi, \xi^{-1}]/(\xi^2 - 1)(\xi - 1)$.

We want to compute $\chi(1)$, and $\chi(\xi)$, $\chi(1+\xi)$.

The naive thing to do is this:

$$\deg \mathrm{ch}(1)\mathrm{Td}(T_{\mathcal{X}}) = \deg[(1)(1 + \frac{3}{2}t)]$$

where we take the coefficient of t, and t is the class of a line, which (because of a stabilizer group) so $\deg t = 1/2$. Thus we get $3/4$.

And then $\mathrm{ch}(\xi)\mathrm{Td}(T_{\mathcal{X}}) = \deg(1+t)(1+\frac{3}{2}t) = \deg\frac{5}{2}t = 5/4$.

Now $\deg \mathrm{ch}(1+\xi)\mathrm{Td}(T_\chi) = 2$, and this is the right answer.

The function $1+\xi$ is supported at 1 (in this ring $K_0(\mathcal{X}) = \mathbb{Z}[\xi,\xi^{-1}]/(\xi^2-1)(\xi-1)$).

$1 = (1+\xi)/2 + (1-\xi)/2$.

We can check this directly:

$\pi : \mathbb{A}^2 - \{0\} \to \mathbb{P}^1$.

$$\pi_* 1 \;=\!=\!=\; \mathcal{O}_{\mathbb{P}^1}$$

$$\pi_* \xi \;=\!=\!=\; \mathcal{O}_{\mathbb{P}^1}$$

$$\pi_* \xi^2 \;=\!=\!=\; \mathcal{O}_{\mathbb{P}^1}(1).$$

$\xi$ corresopnds to the bundle
$$\mathbb{A}^2 \times \mathbb{A}^1 \to \mathbb{A}^2,$$
where the action is $\lambda(x, y, v) = (\lambda x, \lambda^2 y, \lambda v)$.

And 1 corresponds to the bundle:
$$\mathbb{A}^2 \times \mathbb{A}^1 \to \mathbb{A}^2,$$
where the action is $\lambda(x, y, v) = (\lambda x, \lambda^2 y, v)$.

$\mathbb{A}^2 - \{0\}$ is covered by $\operatorname{Spec} k[x, x^{-1}, y]$ and $\operatorname{Spec} k[x, y, y^{-1}]$, and the quotient $\mathbb{P}^1$ is covered by (i) $\operatorname{Spec} k[y/x^2]$, and (ii) $\operatorname{Spec} k[x^2/y]$ respectively.

$\xi$ corresponds to the free $A$-module $M$ generated by $T$, where $T$ has weight 1.

On the first piece, $M^{\mathbb{C}^*}$ is generated by $yT$, On the second piece, $M^{\mathbb{C}^*}$ is generated by ...

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