MATH 245 CLASS 5 (DAN EDIDIN)

RAVI VAKIL

CONTENTS

We start with a correction to the definition of coarse moduli space.

Definition. If $\mathcal{X}$ is a DM stack, a coarse moduli space $\mathcal{X} \to M$ is an algebraic space if

(1) $p : \mathcal{X} \to M$ is the categorical quotient (in category of algebraic spaces). Equivalently, any morphism $\mathcal{X} \to N$ with N an algebraic space factors through a morphism $M \to N$.
(2) Geometric points of the moduli space correspond geometric points of the stack. Translation: we get a homeomorphism of Zariski topologies.

Theorem. If $\mathcal{X}$ has finite stabilizer, then a corase moduli space and $p : \mathcal{X} \to M$ is a geometric quotient ($\pi_* \mathcal{O}_X^G = \mathcal{O}_M$, ie. $p_* \mathcal{O}_X = \mathcal{O}_M$.)

We now define the Inertia Stack for a group quotient.

Suppose G actson X.

Then $I_G X := \{(g, x) : gx = x\} \subset G \times X$. The quotient is $[I_G X/G] = I[X/G]$ which comes with a map to $[X/G]$.

Reason for language: "Inertia" here basically means "stabilizer".

1. LOTS OF WEIGHTED PROJECTIVE SPACES

Given integers $a_0, \dots, a_n$, we can look at $\mathbb{A}^{n+1} \backslash \{0\} / \mathbb{C}^*$, where $\lambda(x_0, \dots, x_n) = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n)$. The quotient stack is $\mathbb{P}(a_1, \dots, a_n)$. (In other words, when we write down this notation, we mean the stack, not the coarse moduli space.)

We will work out the K-theory of this stack.

Consider $\mathbb{P}(a, b)$, which has coarse space $\mathbb{P}^1$.

Date: January 14, 2015.

Now $K_0(\mathbb{P}^1(a,b)) = K_0(\mathbb{C}^*, \mathbb{A}^2 - \{0\})$.

Note: everything at the stack level is smooth, so $K_0(\mathbb{C}^*, \cdot) = G_0(\mathbb{C}^*, \cdot)$.

We have

$$\begin{tikzcd}
K_0(*\mathbb{C}^*, \{0\}) \arrow[r, "i_*"'] \arrow[dr, equal] & K_0(\mathbb{C}^*, \mathbb{A}^2)K_0(\mathbb{C}^8, \mathbb{A}^2 - \{0\} \arrow[r] & 0 \\
& K_0(\mathbb{C}^*, pt) \arrow[u, "p^*", "\cong"'] \arrow[r, equal] & R(\mathbb{C}^*) = \mathbb{Z}[\chi, \chi^{-1}]
\end{tikzcd}$$

(There is a map in the opposite direction to i_* whch is i^*, but I can't latex an arrow back in real time...)

Now $i^* \circ p^* = id$ implies i^* is an isomorhpism.

$i_*(i^*\beta \cdot 1) = \beta \cdot i_* 1$

so $i_* K_0(\mathbb{C}^*, \{0\})$ is principal ideal generated by

$i^* i_* 1 = \lambda_1(N_{\{0\}}(\mathbb{A}^2)^*) = \lambda_{-1}(T_0^* \mathbb{A}^2)$.

We view $\mathbb{A}^2$ as the representation with weights (a, b), so $T_0(\mathbb{A}^2)$ is its class is $\chi^a + \chi^b$. So we have.

$$\lambda_{-1}(T_0^* \mathbb{A}^2) = (1 - \chi^{-a})(1 - \chi^{-b}).$$

Concusion: $K_0(\mathbb{C}^*, \mathbb{A}^2 \setminus 0) = \mathbb{Z}[\chi, \chi^{-1}]/\langle (\chi^a - 1)(\chi^b - 1)$.

Let's now work out the equivariant Chow ring. We do the same thing, basically. $CH_0(\mathbb{C}^*, pt) = \mathbb{Z}[t]$.

Exercise: we get $CH^*(\mathbb{P}(a, b)) = \mathbb{Z}[t]/(at \cdot bt)$. Tensoring this with $\mathbb{C}$, we get $\mathbb{C}[t]/t^2$.

So we see Chow inside K-theory.

More generally, we get $K_0(\mathbb{P}(a_0, \dots, a_n)) = \mathbb{Z}[\chi, \chi^{-1}]/(\chi^{a_0} - 1) \cdots (\chi^{a_n} - 1)$.

And $\chi^*(\mathbb{P}^n(a_0, \dots, a_n)) = \mathbb{Z}[t]/(a_0 \cdots a_n)t^{n+1}$.

Note: $K_0(\mathbb{P}(a_0, \dots, a_n)) = \mathbb{Z}[\chi, \chi^{-1}]/(\chi - 1)^{n+1}$(stuff nonzero at 1).

Riemann-Roch theorem: This is isomorphic to $CH^*(\mathbb{P}(a, b))$, once we localize at 1.

Let's now consider three examples in more detail: $\mathbb{P}(1, 2)$, $\mathbb{P}(1, 3)$, and $\mathbb{P}(1, 2, 4)$.

$K_0(\mathbb{P}(1, 2)) = \mathbb{Q}[\chi]/(\chi - 1)^2 \oplus \mathbb{Q}[\chi]/(\chi + 1)$.

Let's interpret the (-1)-part.

$I\mathbb{P}(1,2) = pI_{\mathbb{C}^*}(\mathbb{A}^2 - \{0\})/\mathbb{C}^*]$.

Let $X = \mathbb{A}^2 - \{0\}$ for convenience.

$I_{\mathbb{C}^*}(\mathbb{A}^2 - \{0\}) = \{(g, x) : gx = x\} \subset \mathbb{C}^* \times X = X^1 \coprod X^{-1} = X \coprod B\mathbb{Z}_2$.

So the (-1)-part is $CH^*([X^{-1}/\mathbb{C}^*])$.

Next let's consider $\mathbb{P}(1,3)$.

$$K_0(\mathbb{P}(1,3)) = \mathbb{Q}[\chi]/((\chi-1)^2(\chi^2+\chi+1)).$$

Now we have $I_{\mathbb{C}^*}X = X^1 \coprod X^{\omega} \coprod X^{\omega^2}$.

Dan wants to tensor with $\mathbb{C}$ to get $\mathbb{C}[\chi]/((\chi-1)^2(\chi-\omega)(\chi-\omega^2))$

Ravi remains unconvinced that the complex numbers are useful.

Next, we consider $\mathbb{P}(1,2,4)$.

$$K_0(\mathbb{P}(1,2,4)) = \mathbb{C}[\chi]/((\chi-1)(\chi^2-1)(\chi^4-1)) = \mathbb{C}[\chi]/(\chi-1)^3(\chi+1)^2(\chi-i)(\chi+i).$$

We thus get $I_G X = X \coprod X^{-1} \coprod X^i \coprod X^{-i}$, so $I\mathbb{P}(1,2,4) = \mathbb{P}(1,2,4) \coprod \mathbb{P}(2,4) \coprod \mathbb{P}(4) \coprod \mathbb{P}(4)$ where $\mathbb{P}(4) = \mu_4$.

Theorem. If G acts on X with finite stabilizers. Then

$$G_0(G,X) = G_0(\mathcal{X})_{\mathbb{C}} \xrightarrow{\sim} CH^*_G(I_G X)_{\mathbb{C}} = CH^*(I\mathcal{X})_{\mathbb{C}}$$

Arnav says: this is Atiyah-Segal-type stuff, which says that the stupid way to do equivariant K-theory is the completion of the smart way to do equivariant K-theory. Also, these things work integrally until later. Ravi is now convinced that right now, we can still work over $\mathbb{Z}$ rather than $\mathbb{C}$.

Thsi follows from localization theorem for equivariant K-theory, and eqvuiariant Riemann-Roch.

We now discuss localization in a way that will be more understandable, not in the most general way.

Suppose $G = T$ s a torus (not necessary, but clearer the first time around.

Suppose $\mathfrak{p}$ is a prime in $R(T)$ (the representation ring).

$\operatorname{Supp} \mathfrak{p}$ is the smallest closed subgroup H such that $\mathfrak{p}$ is in the image of $\operatorname{Spec} R(H)$ (under the canonical map $\operatorname{Spec} R(H) \to \operatorname{Spec} R(T)$ corresponding to $R(T) \to R(H)$).

Localization theorem: Let X^H be the fixed locus. Pushforward $G_0(G, X^H) \to G_0(G, X)$ is an isomorphism after localizing at $\mathfrak{p}$.

If I look at $R(T)_{\mathbb{C}}$, and $\mathfrak{m}_h$ is a maximal ideal of $R(T)_{\mathbb{C}}$, hence corresponds to a point $h \in T$.

Assume $\langle h \rangle$ is finite. Then in this case, the localiation theorem says that $G_0(G, X^h) \to G_0(G, X)$ is an isomorphism after loalizing at $\mathfrak{m}_h$.

Now apply this to:

$$K_0(\mathbb{P}(1,2,4)) = \mathbb{C}[\chi]/(\chi-1)^3 \times \mathbb{C}[\chi]/(\chi+1)^2 \times \mathbb{C}[\chi]/(\chi-i) \times \mathbb{C}[\chi]/(\chi+i).$$

It works in this example!

E-mail address: `vakil@math.stanford.edu`