MATH 245 CLASS 4 (DAN EDIDIN)

RAVI VAKIL

CONTENTS

Suppose G acts on X. We have a quotient stack $[X/G]$.

If X is smooth, then the equivariant Chow ring $CH^*_G(X)$ is a ring of "characteristic classes", by which we mean:

Proposition: It gives a functorial assignment of an operation: $c(E) : CH_*(B) \to CH_*(B)$. for any

$$\begin{tikzcd} E \arrow[r, "\text{G-equiv}"] \arrow[d, "\text{G-principal bdle}"'] & E \\ X & \end{tikzcd}$$

If G acts with finite reduced stabilizers, then $[X/G]$ is a DM stack.

Étale locally, $[X/G]$ is isomorphic to a scheme.

Theorem (Rydh). If $x : \operatorname{Spec} k \to [X/G]$ is a point of the stack (perhaps geometric), then there exists a stabilizer-preserving (= fixed-point reflecting) étale morphism $[U/G_x] \to [X/G]$, where G_x is the automorphism group of x.

1. COURSE MODULI SPACE

Suppose G acts with *finite stabilizer* (not just quasifinite stabilizer) (i.e., $I_G X = \{(g, x) : gx = x\} \to X$ is a finite morphism).

(Aside: E.g. $G = PSL_2$ acts on the open set $\mathbb{P}(\operatorname{Sym}^4 V)$, where V has a defining representation, corresponding to forms with at least 3 distinct roots. It is a very good exercise in GIT. This has quasifinite but not finite stabilizers.)

Date: January 12, 2015.

Then $[X/G]$ has a coarse moduli space M which is a priori only an algebraic space, satisfying the following properties:

(1) $\mathcal{O}_M = \pi_* \mathcal{O}_X^G$.
(2) M is universal for maps $[X/G] \to$ **Alg Sp**.

There is also a proper pushforward.

Theorem (Edidin-Graham). If G acts properly on X, then the proper pushforward (which needs a definition)

$$p_* : CH_*^G(X)_{\mathbb{Q}} \to CH_*(M)_{\mathbb{Q}}$$

is an isomorphism. Moreover, $CH_*^G(X)$ agrees with the "naive" definition: G-invariant cycles modulo invariant rational equivalences.

In particular, if M is complete (i.e., proper over a field) then I can define a degree map (which will be needed for the right side of RR) $\deg CH_0^G(X)_{\mathbb{Q}} \to \mathbb{Q}$.

Note: $\deg[G \cdot x] = 1/|G_x|$ (assuming $k = \overline{k}$).

Now we go to the K-theory side.

Theorem (Edidin-Graham). If G acts with finite stabilizers. Then $G_0(G,X)_{\mathbb{Q}}$ is supported at a finite number of maximal ideals of $\operatorname{Spec} R(G)_{\mathbb{Q}}$. Also know that $CH_G^i(X) = 0$ if $i > \dim X - \dim G$ (so $\prod_{i=0}^{\infty} CH_G^i(X) = CH_G^*(X)$).

So in this case,

$$G_0(G,X)_{\mathbb{Q}} = \oplus_{\mathfrak{m} \in \mathrm{Supp}} G_0(G,X)_{\mathfrak{m}}$$

Augmentation ideal, $\mathfrak{m}_1$. Equivariant RR is now restated as:

$$\begin{tikzcd} G_0(G,X)_{\mathbb{Q}} \arrow[rr, "\tau"] \arrow[dr] & & CH_G^*(X)_{\mathbb{Q}} \\ & G_0(G,X)_{\mathfrak{m}} \arrow[ur, "\sim"'] & \end{tikzcd}$$

To understand the other components of $G_0(G,X)$, we will use the localization theorem in equivariant K-theory.

2. The localization theorem in equivariant K-theory

For the indefinite future, for convenience we assume G is diagonalizble. This isn't necessary, but makes things easier to follow.

Let us do some simple examples, weighted $\mathbb{P}^1$'s.

$\mathbb{P}^1 = (\mathbb{A}^2 - \{0\})/\mathbb{C}^*$.

The K-theory of the left side is the K-theory of the right side. And you know the K-theory of $\mathbb{A}^2$, as it is a vector bundle over a point.

$G(\mathbb{C}^*, \mathbb{A}^2 - \{0\}) = \mathbb{Q}[\chi]/(\chi - 1)^2$. It is a module over the representation ring $\mathbb{Q}[\chi, \chi^{-1}]$, and it is supported at a single point.

$\mathbb{P}(1,2) = (\mathbb{A}^2 - \{0\})/\mathbb{C}^*$ with weights $(1,2)$. So the relation is $(\chi - 1)(\chi^2 - 1)$. We hvae $\mathbb{Q}[\chi]/(\chi-1)^2(\chi+1)$. This module is supported at two points! But if you get rid of the part supported at -1, you get the same thing.

E-mail address: vakil@math.stanford.edu