MATH 245 CLASS 3 (DAN EDIDIN)

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G acts on X, $G \hookrightarrow GL_n(k)$.

We can then define:

$G_0(G, X)$ is the Grothendieck group of coherent sheaves.

$K_0(G, X)$ is the Grothendieck group of vector bundles.

(Thomason '88)

Note: $G_0(G, \operatorname{Spec} k) = K_0(G, \operatorname{Spec} k) = R(G)$ (the representation ring of G).

People knew about equivariant K-theory before they knew about equivariant Chow groups.

The naive way you might want to define $CH^i_G(X)$, by taking equivariant cycles, then it wouldn't satisfy the homotopy axiom — there are some spaces with few G-invariant cycles, but there is a vector bundle over it with many G-invariant cycles.

Exercise: Formulate this naive definition, and prove that it is not a homotopy-invariant.

Burt Totaro observed: If U is an open set in a representation V of G such that G acts freely [to be defined in a later week] on U, and the codimension of $V \setminus U \gg 0$, then $U \to U/G$, and this approximates $EG \to BG$ for Chow theory.

Definition (Edidin-Graham). $CH^k_G(X) := CH^k((X \times U)/G)$ where U is as above, and $\operatorname{codim} V \setminus U > k$.

(We have to show that this is independent of choices.)

Example: $G = \mathbb{G}_m$.

Let $U = \mathbb{A}^{n+1} \setminus \{0\}$, and V is $\mathbb{A}^{n+1}$ where all weights are 1.

Topologists: $B\mathbb{C}^* = \mathbb{CP}^\infty$.

Note: $CH^k_G(X) \neq 0$ for $k > \dim X$. For example:

Exercise: $CH^*_{\mathbb{G}_m}(pt) \cong \mathbb{Z}[t]$.

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Exercise: $CH^*_{\mu_n}(pt) = \mathbb{Z}[t]/(nt)$.

Edidiin-Graham's initial goal: Riemann-Roch (à la Baum-Fulton-MacPherson).

For each (U, V), $U \subset V$ open, G acts freely on U, $\operatorname{codim} V \setminus U \gg 0$.

$$\begin{tikzcd}
G_0(G, X \times V) \arrow[dr, "res"] & G_0((X \times U)/G) \arrow[r, "\sim"] \arrow[d, equal] & CH^*(X \times_G U) & \\
G_0(G, X) \arrow[r] & G_0(G, X \times U) \arrow[r, "\sim"] & CH^*_G(X \times U) \arrow[r, equal, "\text{up to codim } V \setminus U"] & CH^*_G(X)
\end{tikzcd}$$

We take a limit $G_0(G, X) \to \lim_{(U,V)} G_0(G, X \times U) \to \lim_{(U,V)} CH^*_G(X \times U)$.

(Everything is with $\mathbb{Q}$-coefficients, which are suppressed because I=Ravi can't type that fast.)

Now $\lim_{\leftarrow} CH^*_G(X \times U) = \prod_{i=0}^{\infty} CH^i_G(X)_{\mathbb{Q}}$.

It turns out that $\lim_{\leftarrow} G_0(G, X \times U)_{\mathbb{Q}} = G_0\widehat{(G, X)}_{\mathbb{Q}}$ where $\hat{\cdot}$ is completion with respect to either ideal $I_X = \ker(rk : K_0(G, X) \to \mathbb{Q})$ or $I = \ker(rk : R(G) \to \mathbb{Q})$. The first is intrinsic, in terms of the augmentation $K_0([X/G])$, while the latter depends on the presentation.

Equivariant Riemann-Roch is then:

$$\tau^G_X : G_0(G, X) \longrightarrow G_0\widehat{(G, X)}_{\mathbb{Q}} \xrightarrow{\sim} \prod_{i=0}^{\infty} CH^i_G(X)_{\mathbb{Q}}$$

$ch(L) = 1 + c_1 + c_1^2/2 + \cdots$.

Example. Suppose X is a point, $G = \mu_2$.

$G_0(G, pt)$ is the representation ring of μ_2. This is $\mathbb{Q}(\chi)/(\chi^2 - 1)$. This is an Artin ring supported at two points. This is a diagonal group, so over $\mathbb{C}$, we get the coordinate ring of the group.

On the Chow side, $\mathbb{Q}[t]/(2t) = \mathbb{Q}$.

How to match them up?

Answer: the augmentation ideal is $\langle \chi - 1 \rangle$. So the completion is $\mathbb{Q}(\chi)/(\chi - 1)$.

The map is $\chi \to e^t$.

Example. $R(G) = \mathbb{Q}[\chi, \chi^{-1}]$. $\langle \chi - 1 \rangle = \langle \chi^{-1} - 1 \rangle$.

Then $\chi \mapsto e^t$ factors through an isomorphism

$\chi[\chi, \chi^{-1}] = \mathbb{Q}[[(1 - \chi^{-1}]]$.

$CH^*_G(pt) = \mathbb{Q}[t]$.

$\prod CH^i_G(pt) = \mathbb{Q}[[t]]$.

1. 5 MINTUES OF QUOTIENT STACKS

G acts on X, quotient stack $[X/G]$. Without assumptions on action, this is an Artin stack. $X \to [X/G]$ is a principal bundle, and $X \to [X/G]$ is a smooth cover.

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & {[X/G]} \end{tikzcd}$$

$K_0(G, X) = K_0([X/G])$

$G_0(G, X) = G_0([X/G])$

$CH^*([X/G]) = CH^*_G X$.

You can prove that if X is smooth then this ring operations on Chow gruops $CH_*(T)$ for all $T \to [X/G]$

$[X/G]$ is DM if G acts with finite (reduced) stabilizers.

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