Reference: Fulton Chapter 2.
Throughout, $X$ is finite type over a field $k$.

1. HOMOTOPY INVARIANCE OF CHOW GROUPS

If $\pi^* : E \to X$ is a vector bundle of rank $r$, then $\pi^* : CH_* \to CH_{*+r}$ is an isomorphism.

This "fundamental but classical proof" follows from: the projective formula. $p : \mathbb{P}E \to X$, $p_* : CH_*(X) \to CH_*(\mathbb{P}E)$ is a split injection. Better: $CH_*(\mathbb{P}(E)) = CH_*(X)^{\oplus r}$ with basis $1, h, \cdots, h^{r-1}$.

$CH^*(\mathbb{P}(E)) = CH^*(X)[h]/(h^r + c_1(E)h^{r-1} + \cdots + c_r(E))$

Grothendieck defined Chern classes using the coefficients in this relation.

2. CHERN AND SEGRE CLASSES

We start with a simple observation. Suppose $D$ is a Cartier divisor (locally principal). There is an associated Weil divisor:

$$\sum_{V \subset X, \text{codim } V=1} \text{ord}_V(f)[V].$$

(Defining the order of a function along a non-regular codimension 1 Weil divisor requires some care.)

If $D$ is a Cartier divisor, then we have a locally free sheaf of rank 1 $L(D)$.

Conversely, given $L$ a line bundle, there exists a Cartier divisor $D$ such that $L = L(D)$. $D$ is not unique, but $L(D) = L(D')$ iff $D - D'$ is a principal divisor.

Ad hoc definition. $L$ is a line bundle on $X$, and $V \subset X$ is a k-dimensional subvariety. We define $c_1(L) \cap [X] \in CH_{k-1}(V) \to CH_{k-1}(X)$.

This is the class of the Weil divisor associated to a Cartier divisor $D_V$ such that $L|_V = L(D_V)$.

Date: January 8, 2015.
You should think of this as intersecting with a regular embedding of codimension 1.

So we believe that we know how to intersect Cartier divisors with anything.

Let us now think about higher rank bundles.

Suppose $E \to X$ is a rank $r$ vector bundle.

Then $p : \mathbb{P}E \to X$ is a projective bundle over $X$. (Probably $\text{Proj}$ of the symmetric algebra over the dual of $E$.)

Note that $p$ is flat and proper, of relative dimension $r - 1$.

Define operations $s_i(E)$ on $\text{CH}^*(X)$ by formula $s_i(E)(\alpha) = p_*(c_1(O(1))^{i+r-1} \cap p^* \alpha)$

$s_0(E) = 1$,

$E = L$ is a line bundle, $P(E) = X, O(-1) = L$.

$s_i(E) = 1 + s_1(E)t + \cdots + s_r(E)t^r + \cdots$

Then define: $c_t(E) = s_t(E)^{-1} = 1 + c_1(E)t + \cdots + c_r(E)t^r$.

Notice that $c_1 = -s_1, c_2 + s_2 + c_1s_1 = 0$ which implies $c_2 = s_1^2 - s_2$, etc.

Miracle: $c_k = 0$ for $k > \text{rank } E$, regardless of $\text{dim } X$.

(Grothendieck had earlier defined Chern classes in a different way, see the start of this file.)

3. PULLBACKS BY A REGULAR EMBEDDING

Easiest case. $i : D \hookrightarrow X$ is a regular embedding of codimension 1, so it is an effective Cartier divisor.

Then for $\alpha \in \text{CH}^*(X), i^* \alpha = c_1(L(D)) \cap \alpha \in \text{CH}^*(D)$.

$Y \hookrightarrow X$ regular embedding of codimension $d$. We have a normal bundle $\pi : N_{YX} \to Y$ of rank $d$.

Now $\pi^* : \text{CH}^*(Y) \to \text{CH}^{*+d}N_{YX}$ is an isomorphism (by the homotopy property above), so we have an inverse isomorphism $s^* : \text{CH}^*(N_{YX}) \to \text{CH}^{*+d}(Y)$. (This should be interpreted as ”intersecting with the 0 section”.)

Suppose $V \subset X$ is a subvariety of dimension $k$. We now want to define a class $i^*[V] \in \text{CH}^{*+d}(Y)$ (a refined element in $\text{CH}^{*+d}(Y \cap V)$).
Now

\[ W \hookrightarrow V \]
\[ f \]
\[ Y \hookrightarrow X \]

\( W \subset V \) is closed but that’s essentially all I know.

\( C_{W,V} = \text{Spec Sym}(I^k/I^{k+1}) \) where \( I \) is an ideal sheaf of \( W \subset V \). This has pure dimension \( k \), as it is the affine cone over the effective divisor, which is codimension 1.

So now we have \( C_{W,V} \rightarrow W \).

With a little bit of algebra work, we see that we have \( C_{W,V} \hookrightarrow N_YX \). Now we \( s^*[C_{W,V}] \in \text{CH}_{k-d}(Y) \). So that does it!

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