MATH 245 CLASS 2 (DAN EDIDIN)

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CONTENTS

1. CHOW GROUPS

They are like Weil divisors, not Cartier divisors/invertible sheaves.

X finite type over a field.

Define $Z_k(X)$ to be the free abelian group of cycles of k-dimensional sub varieties.

Let $R_{k+1}(X)$ be the subgroup generated by (f) where $f \in K(W)$ for some $(k+1)$-dimensional variety.

$CH_k(X) = Z_k(X)/R_{k+1}(X)$.

For convenience, we define $CH^{n-k}(X) = CH_k(X)$, where $n = \dim X$ (we assume X is pure dimensional).

Key properties:

Pushforwards. $f : X \to Y$ proper, then there is $f_* : CH_k(X) \to CH_k(Y)$.

defined by $f* : [V] = 0$ if $\dim f(V) < \dim V$, and $[V : f(V)][f(V)]$ if $\dim f(V) = \dim V$.

Pullbacks. If $f : X \to Y$ is flat of relative dimension n, then $f^* : CH^k(Y) \to CH^k(X)$, $[Z] \mapsto [f^{-1}(Z)]$.

And if $f :\hookrightarrow Y$ is a regular embedding. $f^* : CH^k(Y) \to CH^*(X)$ given by $[Z] \mapsto$ "$[Z \cap X]$". Making this precise is tricky! It takes place about six chapters into Fulton. (And the core of Fulton is the first eight chapter...)

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If X is smooth separated over k, then $\Delta : X \hookrightarrow X \times X$ is a closed regular embedding, so we can define an intersection product by:

$$[V] \cdot [W] = \Delta^*[V \times W].$$

Coniveau filtration and Chow groups. G_0 has a filtration $G_0 = F \supset \cdots \supset F_n$, $n = \dim X$, where F_k is the subgroup generated by sheaves supported in codimension at least k.

We have a map $CH^i(X) \to F_i/F_{i+1}$.

Chern classes.

Suppose $\mathcal{L}$ is a line bundle. Define an operation $c_1(\mathcal{L}) : CH_k(X) \to CH_{k-1}(X)$.

If $\mathcal{L}$ has lots of global sections, then $c_1(\mathcal{L}) \cap [V] = [D \cap V]$ Then $\mathcal{L}(D) = \mathcal{L}$ and D is transverse to V. (Not easy but in fulton.)

You can extend this to get higher Chern classes. Grothendieck does this using projective bundles. Fulton does this using Segre classes. Here is an idea of how to do this. The splitting principle in algebraic geometry says that you can pull back any vector bundle so that the vector bundle doesn't quite split, but it filters in to sub bundles, so the quotients are line bundles, and then you define the ith Chern class to be the ith elementary symmetric polynomial in the c_1's of the line bundles.

Chern classes live in the "operational" Chow ring. $c_i \in A^i(X)$. A^i is defined to be the group of operations $CH_*X \to CH_{*-i}(X)$ satisfying natural compatibility conditions.

Fact: The map $A^i(X) \to CH^i(X)$ given by $c \mapsto c \cap [X]$ is an isomorphism if X is smooth.

If E is a rank r vector bundle on X, $ch(E) \in A^*(X)_{\mathbb{Q}}$.

$ch(E) \in A^*(X)_{\mathbb{Q}}$ is defined to be $\sum e^{\alpha_i}$ where the α_i are the "Chern roots". This is $r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \cdots$.

We define the Todd class by:

$$td(E) = \prod \frac{\alpha_i}{1 - e^{-\alpha}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \cdots.$$

Note: this is invertible!

Define the *Chern polynomial* of a rank r vector bundle by $c_t(E) = \sum_{i=0}^{r} c_i(E)t^i$.

If we have an exact sequence $0 \to E' \to E \to E'' \to 0$, then $c_t(E) = c_t(E')c_t(E'')$.

Grothendieck-RR.

(a) Suppose X is a nonsingular projective variety, then $ch : K_0(X)_{\mathbb{Q}} \to CH^*(X)_{\mathbb{Q}}$ is an isomorphism.

(b) If $f : X \to Y$ is a morphism of smooth projective varieties (necessarily projective, by the Cancellation Theorem for projective morphisms). Then the following diagram commutes.

$$\begin{array}{ccc} K_0(X) & \xrightarrow{ch \cdot Td(T_X)} & CH^*(X)_{\mathbb{Q}} \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ K_0(Y) & \xrightarrow{ch \cdot Td(T_Y)} & CH^*(Y)_{\mathbb{Q}} \end{array}$$

How would you guess this formula? Look at $\mathbb{P}^n$! Let $h = c_1(\mathcal{O}(1))$. $T_{\mathbb{P}^n}$ fits in to the Euler exact sequence.

$Td(T_X) = Td(\mathcal{O}(1)^{n+1}) = \left(\frac{h}{1-e^{-h}}\right)^{n+1}$.

$ch(\mathcal{O}(n)) = e^{nh}$.

HRR: $\chi(\mathcal{O}(n)) = \deg n$ term in $(e^{nh}) \left(\frac{h}{1-e^{-h}}\right)^{n+1}$.

The genius in GRR is already there in HRR, and that insight seems somehow more believably due to Hirzebruch than to Grothendieck.

If X is smooth and projective, then $ch \cdot Td(T_X')$ is an isomorphism $K_0(X) \to CH^*(X)_{\mathbb{Q}}$.

Baum-Fulton=MacPherson extended this to singular schemes. There exists for all X an isomorphism $T_X : G_0(X) \to CH_*(X)_{\mathbb{Q}}$, which is covariant for proper morphisms. $f : X \to Y$... $f_* T_X(\alpha) = T_X(f_*\alpha)$ if $\beta \in K_0(X)$, $T_X(\beta \cdot \alpha) = ch(B) T_Y(\alpha)$.

2. Group actions

Let's begin to talk about equivariance!

The set-up: G is a linear algebraic group over k, i.e. G is a closed subgroup of $GL_n(k)$.

G acts on X.

We can define G-coherent sheaf, G-vector bundle. (Exercise: what are the diagrams?)

Define $G_0(G, X)$ to be the Grothendieck group of G-coherent sheaves.

Define $K_0(G, X)$ to be the Grothendieck group of G-vector bundles.

These satisfy the same formal properties as K_0 and G_0. Now $K_0(G, pt)$ is the representation ring of G.

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