MATH 210A PROBLEM SET 2

This problem set will be due on Thursday October 16 at noon in Evita Nestoridi's mailbox on the first floor of the mathematics department, or in class. If and when you find typos or errors on this or any other problem set, please let me know, so I can warn others.

1. *(semisimple endormorphisms)* Let V be a finite-dimensional nonzero vector space over a field F. A linear self-map $T : V \to V$ is *semisimple* if every T-stable subspace of V admits a T-stable complementary subspace. (That is, if $T(W) \subseteq W$, then there exists a decomposition $V = W \oplus W'$ with $T(W') \subseteq W'$.) Keep in mind that such a complement is not unique in general (e.g. consider T to be a scalar multiplication with $\dim V > 1$).

(a) For each monic irreducible $\pi \in F[t]$, define $V(\pi)$ to be the subspace of V killed by a power of $\pi(T)$. Prove that $V(\pi) \neq 0$ if and only if π divides the minimal polynomial m_T of T, and that $V = \oplus_{\pi | m_T} V(\pi)$. (In case F is algebraically closed, these are the *generalized eigenspaces* of T on V.)

(b) Use the rational canonical form to prove that T is semisimple if and only if m_T has no repeated irreducible factor over F. (Hint: apply (i) to T-stable subspaces of V to reduce to the case when m_T has one monic irreducible factor.) Deduce that a Jordan block of rank greater than 1 is never semisimple, that m_T is the "square-free part" of χ_T when T is semisimple, and that if $W \subseteq V$ is a T-stable nonzero proper subspace then T is semisimple if and only if the induced endomorphisms $T_W : W \to W$ and $\overline{T} : V/W \to V/W$ are semisimple.

2. *(fun linear algebra)* Suppose V is a 4-dimensional complex vector space, and T is a linear self-map of V with characteristic polynomial $x(x-1)^2(x-2)$. Show that $-T^2 + 2T$ is a projection onto the generalized eigenspace corresponding to 1.

3. *(practice with $\otimes$)* Calculate (with proof!) the following tensor products: (a) $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12)$. (b) $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$.

4. *(right-exactness of $\otimes N$)* Show that $\cdot \otimes_R N$ gives a covariant functor from the category of R-modules to itself. Show that $\cdot \otimes_R N$ is a *right-exact functor*, i.e. if

$$M' \to M \to M'' \to 0$$

is an exact sequence of R-modules (which means $f : M \to M''$ is surjective, and M' surjects onto the kernel of f), then the induced sequence

$$M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

is also exact. (Surjectivity of $M \otimes_R N \to M'' \otimes_R N$ was shown in class.)

Date: October 9, 2014.

5. *(adjointness of $\otimes$ and* Hom*)* Suppose M, N, and P are R-modules. Describe a bijection $\mathrm{Hom}_R(M \otimes_R N, P) \leftrightarrow \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$. If you feel up to it, explain why this bijection is "functorial" or "natural" in both M and P — this is essentially no extra work if you can figure out what it means.

6. *(more linear algebra over a ring)* Show that an $n \times n$ matrix over a ring R is invertible if and only if its determinant is a unit in R.

7. *(practice with $\wedge$)* Suppose M and N are R-modules. Describe (with proof) an isomorphism

$$\bigwedge^n (M \oplus N) \to \bigoplus_{i+j=n} \left(\left(\bigwedge^i M \right) \otimes \left(\bigwedge^j N \right) \right).$$

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