MATH 210A PROBLEM SET 1

This problem set will be due on Thursday October 9 at noon in Evita Nestoridi's mailbox on the first floor of the mathematics department. If and when you find typos or errors on this or any other problem set, please let me know, so I can warn others.

1. (essentially Dummit and Foote Exercises 8.1.9 and 9.4.9 respectively) Let $R = \mathbb{Z}[\sqrt{2}]$.
(a) Show that $\mathbb{Z}[\sqrt{2}]$ is a Euclidean domain, using the norm $N(a + b\sqrt{2}) = |a^2 - 2b^2|$.
(b) Prove that $x^2 - \sqrt{2}$ is irreducible in $R[x]$.

2. Let R be a commutative ring. For R-modules M and N, define the R-module $\mathrm{Hom}_R(M, N)$ to be the set of R-linear maps $M \to N$ endowed with R-linear structure via pointwise operations (i.e. $(T_1 + T_2)(m) = T_1(m) + T_2(m)$, $(r \cdot T)(m) = r \cdot T(m)$).
(a) Check that $\mathrm{Hom}_R(M, N)$ is an R-module. Explain how an R-linear map $L : M' \to M$ (resp. $L : N' \to N$) naturally induces an R-linear map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N)$ (resp. $\mathrm{Hom}_R(M, N') \to \mathrm{Hom}_R(M, N)$).
(b) Taking $N = R$, the *dual* of M is $M^* := \mathrm{Hom}_R(M, R)$. In the case where R is a domain, show that M^* is always torsion-free (even if M is not), and give an example with $R = \mathbb{Z}$ for which an injective map $M' \to M$ has an associated dual map $M^* \to M'^*$ that is not surjective. How does the dual module behave with respect to surjective maps $M' \to M$?

3. Show that $\mathbb{Q}$ is a torsion-free $\mathbb{Z}$-module, but is not a free $\mathbb{Z}$-module.

(Localization of a ring in full generality) A multiplicative subset S of a ring R is a subset closed under multiplication containing 1. We define a ring $S^{-1}R$. The elements of $S^{-1}R$ are of the form a/s where $a \in R$ and $s \in S$, where $a_1/s_1 = a_2/s_2$ if *for some* $s \in S$, $s(s_2a_1 - s_1a_2) = 0$. (This implies that $S^{-1}R$ is the 0-ring if $0 \in S$.) We define $(a_1/s_1) \times (a_2/s_2) = (a_1a_2)/(s_1s_2)$, and $(a_1/s_1) + (a_2/s_2) = (s_2a_1 + s_1a_2)/(s_1s_2)$. You should convince yourself that this construction is well-defined, and indeed defines a ring. We have a canonical map $R \to S^{-1}R$ given by $a \mapsto a/1$.

4.
(a) Show that $R \to S^{-1}R$ is injective if and only if S contains no zero-divisors. (A *zero-divisor* of a ring R is an element a such that there is a non-zero element b with $ab = 0$. The other elements of R are called *non-zero-divisors*. For example, a unit is never a zero-divisor.)
(b) Describe a bijection between the primes of $S^{-1}R$ and those primes of R not meeting S.

(Localization of a module in full generality) If M is an R-module, we define the localization $S^{-1}M$ as follows; $S^{-1}M$ will be an $S^{-1}R$-module. The elements of $S^{-1}M$ are of the form m/s where $m \in M$ and $s \in S$, where $m_1/s_1 = m_2/s_2$ if *for some* $s \in S$, $s(s_2m_1 - s_1m_2) = 0$. We define the abelian group structure by $(m_1/s_1) + (m_2/s_2) = (s_2m_1 + s_1m_2)/(s_1s_2)$, and the $S^{-1}R$-action by $(a_1/s_1) \times (m_2/s_2) = (a_1m_2)/(s_1s_2)$. You should convince yourself that

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this construction is well-defined, and indeed defines a $S^{-1}R$-module. We have a canonical map $M \to S^{-1}M$ given by $m \mapsto m/1$.

5. Recall the definition of the *rank* of a module M over an integral domain R, as the maximum number of R-linearly independent elements of M. Let $S = R \setminus \{0\}$, so that $K = S^{-1}R$ is the field of fractions of R. Show that the rank of an R-module M is the dimension over K of $S^{-1}M$.

6. Continuing the notation of the previous problem, show that M is torsion if and only if it has rank 0.

7. (This is essentially problem 12.1.3 in Dummit and Foote.) Show that the rank is additive under direct sums.

8. (Dummit and Foote 12.1.15 — there is also a hint there if you need it.) Prove that if R is a Noetherian ring, then R^n is a Noetherian R-module.

E-mail address: `vakil@math.stanford.edu`