MATH 210A PROBLEM SET 4

This problem set will be due on Friday, October 15, 2010 by 3 pm in Jeremy Miller’s mailbox. You can hand it in during class as well. Let me know of any typos or errors, so I can warn others.

1. Let $V$ be a finite-dimensional nonzero vector space over a field $F$. A linear self-map $T : V \to V$ is semisimple if every $T$-stable subspace of $V$ admits a $T$-stable complementary subspace. (That is, if $T(W) \subseteq W$, then there exists a decomposition $V = W \oplus W'$ with $T(W') \subseteq W'$.) Keep in mind that such a complement is not unique in general (e.g. consider $T$ to be a scalar multiplication with $\dim V > 1$).

   (a) For each monic irreducible $\pi \in F[t]$, define $V(\pi)$ to be the subspace of $V$ killed by a power of $\pi(T)$. Prove that $V(\pi) \neq 0$ if and only if $\pi$ divides the minimal polynomial $m_T$ of $T$, and that $V = \bigoplus \pi | m_T V(\pi)$. (In case $F$ is algebraically closed, these are the generalized eigenspaces of $T$ on $V$.)

   (b) Use the rational canonical form to prove that $T$ is semisimple if and only if $m_T$ has no repeated irreducible factor over $F$. (Hint: apply (i) to $T$-stable subspaces of $V$ to reduce to the case when $m_T$ has one monic irreducible factor.) Deduce that a Jordan block of rank greater than 1 is never semisimple, that $m_T$ is the “square-free part” of $\chi_T$ when $T$ is semisimple, and that if $W \subseteq V$ is a $T$-stable nonzero proper subspace then $T$ is semisimple if and only if the induced endomorphisms $T_W : W \to W$ and $\bar{T} : V/W \to V/W$ are semisimple.

2. Suppose $V$ is a 4-dimensional complex vector space, and $T$ is a linear self-map of $V$ with characteristic polynomial $x(x - 1)^2(x - 2)$. Show that $-T^2 + 2T$ is a projection onto the generalized eigenspace corresponding to 1.

3. Let $V \neq 0$ be a finite-dimensional vector space over a field $F$, and let $T : V \to V$ be a linear self-map. Let $n = \dim V$.

   (a) Prove the following are equivalent:

   (i) $T^N = 0$ for some $N \geq 1$.

   (ii) $T^n = 0$.

   (iii) With respect to some ordered basis of $V$, the matrix for $T$ is upper triangular with 0’s on the diagonal.

   (iv) The characteristic polynomial is $x^n$.

If these conditions hold, we call $T$ nilpotent.

   (b) We say that $T$ is unipotent if $T - 1$ is nilpotent. Formulate characterizations of unipotency analogous to the conditions in (a), and prove that every unipotent $T$ is invertible.

   (c) Assume $F$ is algebraically closed. Using Jordan canonical form, prove that there is a unique expression $T = T_{ss} + T_n$ where $T_{ss}$ and $T_n$ are a pair of commuting endomorphisms of $V$ with $T_{ss}$ semisimple and $T_n$ nilpotent. (This is the additive Jordan decomposition of $T$.)
Show by example with \( \text{dim } V = 2 \) that uniqueness fails if we drop the “commuting” requirement, and show in general that the characteristic polynomial of \( T \) is the characteristic polynomial of \( T_{ss} \) (so \( T \) is invertible if and only if \( T_{ss} \) is invertible).

4. Verify that \( A \to S^{-1}A \) satisfies the following universal property: \( S^{-1}A \) is initial among \( A \)-algebras \( B \) where every element of \( S \) is sent to a unit in \( B \). (Recall: the data of “an \( A \)-algebra \( B \)” and “a ring map \( A \to B \)” the same.) Translation: any map \( A \to B \) where every element of \( S \) is sent to a unit must factor uniquely through \( A \to S^{-1}A \).

E-mail address: vakil@math.stanford.edu