MATH 210A PROBLEM SET 4

This problem set will be due on Friday, October 15, 2010 by 3 pm in Jeremy Miller's mailbox. You can hand it in during class as well. Let me know of any typos or errors, so I can warn others.

1. Let V be a finite-dimensional nonzero vector space over a field F. A linear self-map $T : V \to V$ is *semisimple* if every T-stable subspace of V admits a T-stable complementary subspace. (That is, if $T(W) \subseteq W$, then there exists a decomposition $V = W \oplus W'$ with $T(W') \subseteq W'$.) Keep in mind that such a complement is not unique in general (e.g. consider T to be a scalar multiplication with $\dim V > 1$).

(a) For each monic irreducible $\pi \in F[t]$, define $V(\pi)$ to be the subspace of V killed by a power of $\pi(T)$. Prove that $V(\pi) \neq 0$ if and only if π divides the minimal polynomial m_T of T, and that $V = \oplus_{\pi | m_T} V(\pi)$. (In case F is algebraically closed, these are the *generalized eigenspaces* of T on V.)

(b) Use the rational canonical form to prove that T is semisimple if and only if m_T has no repeated irreducible factor over F. (Hint: apply (i) to T-stable subspaces of V to reduce to the case when m_T has one monic irreducible factor.) Deduce that a Jordan block of rank greater than 1 is never semisimple, that m_T is the "square-free part" of χ_T when T is semisimple, and that if $W \subseteq V$ is a T-stable nonzero proper subspace then T is semisimple if and only if the induced endomorphisms $T_W : W \to W$ and $\overline{T} : V/W \to V/W$ are semisimple.

2. Suppose V is a 4-dimensional complex vector space, and T is a linear self-map of V with characteristic polynomial $x(x-1)^2(x-2)$. Show that $-T^2 + 2T$ is a projection onto the generalized eigenspace corresponding to 1.

3. Let $V \neq 0$ be a finite-dimensional vector space over a field F, and let $T : V \to V$ be a linear self-map. Let $n = \dim V$.

(a) Prove the following are equivalent:

(i) $T^N = 0$ for some $N \geq 1$.
(ii) $T^n = 0$.
(iii) With respect to some ordered basis of V, the matrix for T is upper triangular with 0's on the diagonal.
(iv) The characteristic polynomial is x^n.

If these conditions hold, we call T *nilpotent*.

(b) We say that T is *unipotent* if $T - 1$ is nilpotent. Formulate characterizations of unipotence analogous to the conditions in (a), and prove that every unipotent T is invertible.

(c) Assume F is algebraically closed. Using Jordan canonical form, prove that there is a unique expression $T = T_{ss} + T_n$ where T_{ss} and T_n are a pair of commuting endomorphisms of V with T_{ss} semisimple and T_n nilpotent. (This is the *additive Jordan decomposition* of T.)

Show by example with $\dim V = 2$ that uniqueness fails if we drop the "commuting" requirement, and show in general that the characteristic polynomial of T is the characteristic polynomial of T_{ss} (so T is invertible if and only if T_{ss} is invertible).

4. Verify that $A \to S^{-1}A$ satisfies the following universal property: $S^{-1}A$ is initial among A-algebras B where every element of S is sent to a unit in B. (Recall: the data of "an A-algebra B" and "a ring map $A \to B$" the same.) Translation: any map $A \to B$ where every element of S is sent to a unit must factor uniquely through $A \to S^{-1}A$.

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