MATH 210A PROBLEM SET 3

This problem set will be due on Friday, October 8, 2010 by 3 pm in Jeremy Miller's mailbox. You can hand it in during class as well. Let me know of any typos or errors, so I can warn others.

1. Let k be any field. Show that $I = (wy - x^2, xz - y^2, wz - xy) \subset k[w, x, y, z]$ is a prime ideal by showing that $k[w, x, y, z]/I$ is isomorphic to the subring of $k[a, b]$ generated by monomials of degree divisible by 3. (Hint: think: $(w, x, y, z) = (a^3, a^2b, ab^2, b^3)$.)

2. Let R be a commutative ring. For R-modules M and N, define the R-module $\mathrm{Hom}_R(M, N)$ to be the set of R-linear maps $M \to N$ endowed with R-linear structure via pointwise operations (i.e. $(T_1 + T_2)(m) = T_1(m) + T_2(m)$, $(r \cdot T)(m) = r \cdot T(m)$).
(i) Check that $\mathrm{Hom}_R(M, N)$ is an R-module. Explain how an R-linear map $L : M' \to M$ (resp. $L : N' \to N$) naturally induces an R-linear map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N)$ (rsep. $\mathrm{Hom}_R(M, N') \to \mathrm{Hom}_R(M, N)$).
(ii) Taking $N = R$, the *dual* of M is $M^* := \mathrm{Hom}_R(M, R)$. In the case where R is a domain, show that M^* is always torsion-free (even if M is not), and give an example with $R = \mathbb{Z}$ for which an injective map $M' \to M$ has an associated dual map $M^* \to M'^*$ that is not surjective. How does the dual module behave with respect to surjective maps $M' \to M$?

3. Show that $\mathbb{Q}$ is a torsion-free $\mathbb{Z}$-module, but is not a free $\mathbb{Z}$-module.

(Localization of a module in full generality) A multiplicative subset S of a ring R is a subset closed under multiplication containing 1. In problem set 2, we defined a ring $S^{-1}R$. If M is an R-module, we define the localization $S^{-1}M$ similarly; $S^{-1}M$ will be an $S^{-1}R$-module. The elements of $S^{-1}M$ are of the form m/s where $m \in M$ and $s \in S$, where $m_1/s_1 = m_2/s_2$ if *for some* $s \in S$, $s(s_2m_1 - s_1m_2) = 0$. We define the abelian group structure by $(m_1/s_1) + (m_2/s_2) = (s_2m_1 + s_1m_2)/(s_1s_2)$, and the $S^{-1}R$-action by $(a_1/s_1) \times (m_2/s_2) = (a_1m_2)/(s_1s_2)$. You should convince yourself that this construction is well-defined, and indeed defines a $S^{-1}R$-module. We have a canonical map $M \to S^{-1}M$ given by $m \mapsto m/1$.

4. Recall the definition of the *rank* of a module M over an integral domain R, as the maximum number of R-linearly independent elements of M. Let $S = R \setminus \{0\}$, so that $K = S^{-1}R$ is the field of fractions of R. Show that the rank of an R-module M is the dimension over K of $S^{-1}M$.

5. (This is essentially problem 12.1.3 in Dummit and Foote.) Show that the rank is additive under direct sums.

6. (Dummit and Foote 12.1.15 — there is also a hint there if you need it.) Prove that if R is a Noetherian ring, then R^n is a Noetherian R-module.

Date: October 5, 2010 revision.

7. Read the statement and proof of the Chinese Remainder Theorem for commutative rings. As an application, prove that if $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are pairwise distinct maximal ideals in a commutative ring R, and $e_1, \dots, e_n \geq 1$, then the natural map $R/(\cap \mathfrak{m}_i^{e_i}) \to \prod (R/\mathfrak{m}_i^{e_i})$ is an isomorphism. (Taking $R = \mathbb{Z}$ and $\mathfrak{m}_i = p_i\mathbb{Z}$ for distinct positive primes p_i recovers the classical Chinese Remainder Theorem.)

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