This problem set will be due on Friday, October 1, 2010 by 3 pm in Jeremy Miller’s mailbox. You can hand it in during class as well. Let me know of any typos or errors, so I can warn others.

1. Suppose \( R \) is an integral domain. Show that \( R[[x]] \) is an integral domain. (Here \( R[[x]] \) is the ring of formal power series over \( R \), see Lang p. 205 or Dummit and Foote p. 238.)

2. Suppose \( p \) is a prime ideal of \( R \). Give a bijection between the prime ideals of \( R \) containing \( p \), and the prime ideals of \( R/p \).

3. Suppose \( \omega \) is a primitive cube root of 1. Show that \( \mathbb{Z}[[\omega]] \) is a unique factorization domain. (Possible hint: look up the proof that \( \mathbb{Z}[[i]] \) is a euclidean domain in Dummit and Foote 8.1.) Factor into primes in \( \mathbb{Z}[[\omega]]: 2, 3, 5, 7 \).

4. (essentially Dummit and Foote Exercises 8.1.9 and 9.4.9 respectively) Let \( R = \mathbb{Z}[\sqrt{2}] \).
   (a) Show that \( \mathbb{Z}[\sqrt{2}] \) is a Euclidean domain, using the norm \( N(a + b\sqrt{2}) = |a^2 - 2b^2| \).
   (b) Prove that \( x^2 - \sqrt{2} \) is irreducible in \( R[x] \).

5. Suppose \( R \) is a unique factorization domain. Show that any localization of \( S \) is a unique factorization domain.

   (Localization of a ring in full generality) A multiplicative subset \( S \) of a ring \( R \) is a subset closed under multiplication containing 1. We define a ring \( S^{-1}R \). The elements of \( S^{-1}R \) are of the form \( a/s \) where \( a \in R \) and \( s \in S \), where \( a_1/s_1 = a_2/s_2 \) if for some \( s \in S \), \( s(s_2a_1 - s_1a_2) = 0 \). (This implies that \( S^{-1}R \) is the \( 0 \)-ring if \( 0 \in S \)) We define \((a_1/s_1) \times (a_2/s_2) = (a_1a_2)/(s_1s_2)\), and \((a_1/s_1) + (a_2/s_2) = (s_2a_1 + s_1a_2)/(s_1s_2)\). You should convince yourself that this construction is well-defined, and indeed defines a ring. We have a canonical map \( R \to S^{-1}R \) given by \( a \mapsto a/1 \).

6. Show that \( R \to S^{-1}R \) is injective if and only if \( S \) contains no zero-divisors. (A zero-divisor of a ring \( R \) is an element \( a \) such that there is a non-zero element \( b \) with \( ab = 0 \). The other elements of \( R \) are called non-zero-divisors. For example, a unit is never a zero-divisor.)

7. Describe a bijection between the primes of \( S^{-1}R \) and those primes of \( R \) not meeting \( S \).

8. Suppose \( R \) is a ring, with a prime \( p \) and a multiplicative set \( S \). (a) Is there a bijection between those maximal ideals of \( R \) which \( p \) and the maximal ideals of \( R/p \)?
   (b) Is there a bijection between those maximal ideals of \( R \) which don’t meet \( S \) and the maximal ideals of \( S^{-1}R \)? (Hint: what if \( R \) is an integral domain, and \( S = R \setminus \{0\} \))
   (In each case, justify your answer with an argument or counterexample.)

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