MATH 210A PROBLEM SET 2

This problem set will be due on Friday, October 1, 2010 by 3 pm in Jeremy Miller's mailbox. You can hand it in during class as well. Let me know of any typos or errors, so I can warn others.

1. Suppose R is an integral domain. Show that $R[[x]]$ is an integral domain. (Here $R[[x]]$ is the ring of formal power series over R, see Lang p. 205 or Dummit and Foote p. 238.)

2. Suppose $\mathfrak{p}$ is a prime ideal of R. Give a bijection between the prime ideals of R containing $\mathfrak{p}$, and the prime ideals of $R/\mathfrak{p}$.

3. Suppose ω is a primitive cube root of 1. Show that $\mathbb{Z}[\omega]$ is a unique factorization domain. (Possible hint: look up the proof that $\mathbb{Z}[i]$ is a euclidean domain in Dummit and Foote 8.1.) Factor into primes in $\mathbb{Z}[\omega]$: 2, 3, 5, 7.

4. (essentially Dummit and Foote Exercises 8.1.9 and 9.4.9 respectively) Let $R = \mathbb{Z}[\sqrt{2}]$.
(a) Show that $\mathbb{Z}[\sqrt{2}]$ is a Euclidean domain, using the norm $N(a + b\sqrt{2}) = |a^2 - 2b^2|$.
(b) Prove that $x^2 - \sqrt{2}$ is irreducible in $R[x]$.

5. Suppose R is a unique factorization domain. Show that any localization of S is a unique factorization domain.

(Localization of a ring in full generality) A multiplicative subset S of a ring R is a subset closed under multiplication containing 1. We define a ring $S^{-1}R$. The elements of $S^{-1}R$ are of the form a/s where $a \in R$ and $s \in S$, where $a_1/s_1 = a_2/s_2$ if *for some* $s \in S$, $s(s_2a_1 - s_1a_2) = 0$. (This implies that $S^{-1}R$ is the 0-ring if $0 \in S$.) We define $(a_1/s_1) \times (a_2/s_2) = (a_1a_2)/(s_1s_2)$, and $(a_1/s_1)+(a_2/s_2) = (s_2a_1+s_1a_2)/(s_1s_2)$. You should convince yourself that this construction is well-defined, and indeed defines a ring. We have a canonical map $R \to S^{-1}R$ given by $a \mapsto a/1$.

6. Show that $R \to S^{-1}R$ is injective if and only if S contains no zero-divisors. (A *zero-divisor* of a ring R is an element a such that there is a non-zero element b with $ab = 0$. The other elements of R are called *non-zero-divisors*. For example, a unit is never a zero-divisor.)

7. Describe a bijection between the primes of $S^{-1}R$ and those primes of R not meeting S.

8. Suppose R is a ring, with a prime $\mathfrak{p}$ and a multiplicative set S. (a) Is there a bijection between those maximal ideals of R which $\mathfrak{p}$ and the maximal ideals of $R/\mathfrak{p}$?
(b) Is there a bijection between those maximal ideals of R which don't meet S and the maximal ideals of $S^{-1}R$? (Hint: what if R is an integral domain, and $S = R \setminus \{0\}$?)
(In each case, justify your answer with an argument or counterexample.)

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