MATH 210A PROBLEM SET 1

This problem set will be due on Friday September 24 at a time and location to be announced on the website by Wednesday, but certainly after class. (You can hand it in during class as well.) If and when you find typos or errors on this or any other problem set, please let me know, so I can warn others.

1. Let R be an associative ring. For $n \geq 1$, rigorously define the *polynomial ring* $R[X_1, \ldots, X_n]$ as follows. The set $R[X_i]_{i \in I}$ consists of functions $f : (\mathbb{Z}^{\geq 0})^n \to R$ that vanish at all but finitely many elements of $(\mathbb{Z}^{\geq 0})^n$. Loosely speaking, f corresponds to $\sum f(J)X^J$ (as J varies through $(\mathbb{Z}^{\geq 0})^n$ — here $X^J := X_1^{j_1} \cdots X_n^{j_n}$ for $J = (j_1, \ldots, j_n)$).

(i) Define an R-module structure via pointwise operations on f, and define the *product* $(f \cdot g)(J) = \sum_{J'+J''=J} f(J')g(J'')$. (This is a finite sum. Some of you may enjoy thinking about it as a convolution.) Show that $(f \cdot g)(J) = 0$ for all but finitely many J, and that this makes $R[X_1, \ldots, X_n]$ into an associative ring containing R as a subring.

(ii) Define X_j to be the function $(\mathbb{Z}^{\geq} 0)^n \to R$ vanishing away fro $(0, \ldots, 1, \ldots, 0)$ (all 0's except for a 1 in the jth slot), which it carries to 1. Prove that the X_j's are in the center of $R[X_1, \ldots, X_n]$ and that each $f \in R[X_1, \ldots, X_n]$ has a unique expression as a finite sum $\sum a_J X^J$ with $a_J \in R$.

(Remark: you can also prove the following "universal mapping property": if $\phi : R \to A$ is a map of associative rings and $a_1, \ldots, a_n \in A$ commute with each other and with $\phi(R)$ then there is a unique ring map $R[X_1, \ldots, X_n] \to A$ extending ϕ and satisfying $X_i \mapsto a_i$ for all i. Universal mapping properties will come up repeatedly in this class.)

2. *(Leveraging linear algebra over a field to linear algebra over a general ring.)* Let R be a commutative ring. The set $M_n(R)$ of $n \times n$ matrices with entries in R has an associative R-algebra structure given by the usual formulas. Note that if $R' \to R$ is a map of commutative rings then applying it on matrix entries defines a map $M_n(R') \to M_n(R)$ of associative rings.

(i) Define the determinant $\det : M_n(R) \to R$ by the usual formula (as a sum indexed by the symmetric group S_n). Using the theory of determinants over a field, show that det is multiplicative when R is any domain. Then for any $m = r_{ij}$ and $m' = (r'_{ij})$ in $M_n(R)$, use the unique ring map $\mathbb{Z}[x_{ij}, x'_{ij}] \to R$ satisfying $x_{ij} \mapsto r_{ij}$ and $x'_{ij} \mapsto r'_{ij}$ to deduce the multiplicativity of det in general (by reduction to the case of the ring $\mathbb{Z}[x_{ij}, x'_{ij}]$ that is an integral domain!).

(ii) Using the same technique, prove the Cayley-Hamilton theorem in $M_n(R)$ for any R (by reducing it to the case over an algebraically closed field, which you are assumed to have seen before).

(iii) Define the trace $\mathrm{Tr} : M_n(R) \to R$ by the usual formula $(r_{ij}) \mapsto \sum r_{ii}$. Prove that $\mathrm{Tr}(mm') = \mathrm{Tr}(m'm)$ by reducing it to the case over a field (which I'll assume you know).

(iv) Prove Cramer's formula over any commutative ring (again reducing to the known case over a field), so in particular $m \in M_n(R)^\times$ if and only if $\det(m) \in R^\times$.

Date: September 20, 2010.

3. *(basically, Lang p. 115 problem 10)* Let D be a positive integer, and let R be those complex numbers of the form $a + b\sqrt{-D}$ with $a, b \in \mathbb{Z}$.

(i) Show that R is a ring.

(ii) Using the fact that complex conjugation is an automorphism of $\mathbb{C}$, show that complex conjugation induces an automorphism of R.

(iii) Show that if $D \geq 2$, then the only units in R are ± 1.

(iv) Show that 3, $2+\sqrt{-5}$, $2-\sqrt{-5}$ are irreducible elements in $\mathbb{Z}[\sqrt{-5}]$. As $3^2 = (2+\sqrt{-5})(2-\sqrt{-5})$, this shows that $\mathbb{Z}[\sqrt{-5}]$ does not have unique factorization.

4. *(basically Lang p. 115 problem 11)* Let R be the functions $\mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx),$$

where the a_i and b_j are real numbers. Show that this is a ring. (This is called the *ring of trigonometric polynomials*.) Define the **trigonometric degree** $\deg_{tr}(f)$ to be the maximum of the integers r, s such that $a_r, b_s \neq 0$. Prove that $\deg_{tr}(fg) = \deg_{tr}(f) + \deg_{tr}(g)$. Deduce from this that R has no divisors of 0 (other than 0), and also deduce that the functions $\sin x$ and $1 - \cos x$ are irreducible elements in the ring.

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