PETER'S ARGUMENT FOR THE FUNDAMENTAL THEOREM OF FINITE ABELIAN GROUPS

The hard part of the fundamental theorem of finite abelian groups is showing that every finite abelian group can be expressed as a product of cyclic groups.

We proceed by the induction on the size of the abelian group. Suppose G is an abelian group. Suppose g_0 is an element of G of maximal order, say m.

Exercise 1. If g is any element of G, show that its order m must divide m_0.

Let $G_0 = \langle g_0 \rangle$. By the inductive hypothesis, we may write $G/G_0 \cong \mathbb{Z}/m_1 \times \cdots \times \mathbb{Z}/m_r$. Choose lifts $g_1, \ldots, g_r$ from generators of the r direct summands.

Exercise 2. Show that g_i has order divisible by m_i.

Exercise 3. (a) Let $G_+ = \langle g_1, \ldots, g_r \rangle$. Show that $G_+ \to G/G_0$ is a surjection.
(b) If g_i had order m_i (for $1 \leq i \leq r$), show that $G_+ \to G/G_0$ is an *isomorphism*. In this case, show that the map $\mathbb{Z}/m_0 \times \mathbb{Z}/m_1 \times \cdots \times \mathbb{Z}/m_r \to G$ given by $(j_0, \ldots, j_m) \mapsto g_0^{j_0} \cdots g_m^{j_m}$ is an isomorphism, so G is indeed the product of cyclic groups.

Unfortunately, the hypotheses of Exercise 3(b) may not be satisfied. (Possible exercise: give a counterexample.) So we patch this problem.

Exercise 4. Show that $g_i^{m_i} = g_0^{s_i}$ for some $s_i \in \mathbb{Z}$. Choose s_i so that it is the smallest nonnegative such integer. (If $s_i = 0$ for all i, we are in the wonderful case where the hypotheses of Exercise 3(b) is satisfied.)

Since g_i has order dividing m_0 (Exercise 1), $g_i^{m_0} = e$, from which $g_0^{s_i m_0/m_i} = e$, from which $m_i | s_i$. Let $t_i = -s_i/m_i$, so that $(g_i g_0^{t_i}) = e$.

If we replace g_i by $g_i g_0^{t_i}$, then we are in the situation of Exercise 3(b), so we win.

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