PROPERTIES OF GEOMETRIC FIBERS

1. Introduction

This is an addendum to the discussion in Math 216 in November 2009 on properties of geometric fibers. It was motivated by discussions with Greg Brumfiel, and was massively improved by Brian Conrad. The discussion doesn't naturally fit into the course, as it is of interest to a smaller than usual minority of those present, and involves different methods. But the arguments are worth thinking through, because (i) many expositions in the literature for historical reasons require quoting old commutative algebra texts, and (ii) the actual arguments are fun, and introduce useful tricks turning questions about general schemes over a field to questions about finite type schemes.

For something to consult (in general), I recommend [Stacks]. If you are interested in a particular fact, and want to know why it is true, without having to read hundreds of pages in advance, I find that you can do this with [Stacks], much as you can in commutative algebra with Eisenbud's book.

If K is a field, then X_K will denote a scheme over K. If K/k is a field extension, X_K will denote $X_k \times_k K$ (where as usual, K sometimes sloppily denotes $\operatorname{Spec} K$). Recall that a k-scheme X_k is *geometrically connected* (resp. *geometrically irreducible, geometrically integral, geometrically reduced*) if for every algebraically closed K/k, X_K is connected (resp. irreducible, integral, reduced).

The facts I wish to prove are Corollary 3.5 on connectedness, Proposition 4.4 on irreducibility, Proposition 5.5 on reducedness, and Theorem 6.1 on varieties.

This discussion is intended to be self-contained within the context of where we are in Math 216, except for the following. I assume you've seen some commutative algebra background facts. If you prefer, replace "separably closed" and "perfect" with "algebraically closed" throughout; you will lose little. There is one fact I won't prove, that could reasonably be proved here:

1.1. *Proposition. — Suppose* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$ *are finite type* k*-algebras. Then* $\operatorname{Spec} A \otimes_k \operatorname{Spec} B \to \operatorname{Spec} B$ *is an open map.*

We will later prove that any flat morphism locally of finite type is open. You could prove this by hand, but I'd rather you not worry about this now. (Here is the beginning of a sketch. It suffices to show that the image of a distinguished open $D(f)$, where $f \in A \otimes_k B$, is open in $\operatorname{Spec} B$. Reduce to the case $A = k[a_1, \dots, a_m]$ (if $A = k[a_1, \dots, a_m]/I$, lift $f \in B[a_1, \dots, a_n]/IB$ to $B[a_1, \dots, a_m]$). Write $B = k[x_1, \dots, x_n]/J$, and lift f to an element

Date: January 3, 2010, corrected Oct. 18, 2010.

of $k[a_1, \ldots, a_m][x_1, \ldots, x_n]$. Interpret f as a family of polynomials in the x_j with variable coefficients. Show that the condition for f to lie in J is an algebraic condition on the coefficients of the monomials in the x_i. Interpret this as the desired result. This idea may be easier to complete after reading this entire discussion.)

1.2. Unrelated remark from Brian Conrad. Here is a sign that things like "geometrically irreducible fibers" is a good property. Show that a polynomial over $\mathbb{Z}$ which is irreducible over $\overline{\mathbb{Q}}$ is irreducible over $\overline{\mathbb{F}_p}$ for all but finitely many p, but that this isn't the case if $\overline{\mathbb{F}_p}$ is replaced by $\mathbb{F}_p$. (More generally, if for a flat morphism locally of finite presentation, the locus of points in the target where the geometric fibers are reduced is open, and similarly with "reduced" is replaced by "integral", see EGA IV_3.12.1.1. Many other similar properties hold too.)

2. Preliminary discussion

2.1. *Lemma* [Stacks, 0383]. — *Suppose X is a k-scheme. Then $X \to \operatorname{Spec} k$ is universally open, i.e. remains open after any base change.*

Proof. If S is an arbitrary k-scheme, we wish to show that $X_S \to S$ is open. It suffices to consider the case $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} B$. To show that $\phi : \operatorname{Spec} A \otimes_k B \to \operatorname{Spec} B$ is open, it suffices to show that the image of a distinguished open set $D(f)$ ($f \in A \otimes_k B$) is open.

We come to a trick we will use repeatedly, which I'll call the tensor-finiteness trick. Write $f = \sum a_i \otimes b_i$, where the sum is *finite*. It suffices to replace A by the subring generated by the a_i. (Reason: if this ring is A', then factor ϕ through $\operatorname{Spec} A' \otimes_k B$.) Thus we may assume A is finitely generated over k. Then use Theorem 1.1. $\square$

2.2. *Lemma.* — *Suppose E/F is purely inseparable (i.e. any $a \in E$ has minimal polynomial over F with only one root, perhaps with multiplicity). Suppose X is any F-scheme. Then $\phi : X_E \to X$ is a homeomorphism.*

Proof. The morphism ϕ is a bijection, so we may identify the points of X and X_E. (Reason: for any point $p \in X$, the scheme-theoretic fiber $\phi^{-1}(p)$ is a single point, by the definition of pure inseparability.) The morphism ϕ is continuous (so opens in X are open in X_E), and by Lemma 2.1, ϕ is open (so opens in X are open in X_E). $\square$

2.3. Exercise. Suppose E/F is a purely inseparable extension. Show that $\mathrm{pr}_2 : \operatorname{Spec} E \otimes_F E \to \operatorname{Spec} E$ is a homeomorphism. (Hint: show it is a bijection, then argue as in Lemma 2.2.) Hence the diagonal map $\delta : \operatorname{Spec} E \to \operatorname{Spec} E \otimes_F E$, which is a section of pr_2, is also a homeomorphism.

3. CONNECTEDNESS

Recall that a connected component of a topological space is a maximal connected subset. (Then one can easily check that every point is contained in a connected component, and connected components are always closed [Stacks, 004T].)

3.1. Topological exercise. Suppose $\phi : X \to Y$ is open, and has non-empty connected fibers. Then ϕ induces a bijection of connected components.

3.2. *Lemma. — Suppose* X *is geometrically connected over* k*. Then for any scheme* Y/k*,* $X \times_k Y \to Y$ *induces a bijection of connected components.*

Proof. Combine Lemma 2.1 and Exercise 3.1. $\square$

3.3. Exercise. Show that a scheme X is disconnected if and only if there exists a function $e \in \Gamma(X, \mathcal{O}_X)$ that is an idempotent ($e^2 = e$) distinct from 0 and 1. (Hint: if X is the disjoint union of two open sets X_0 and X_1, let e be the function that is 0 on X_0 and 1 on X_1. Conversely, given such an idempotent, define $X_0 = V(e)$ and $X_1 = V(1-e)$.)

3.4. *Proposition. — Suppose* k *is separably closed, and* A *is an* k*-algebra with* $\operatorname{Spec} A$ *connected. Then* $\operatorname{Spec} A$ *is geometrically connected over* k*.*

Proof. We wish to show that $\operatorname{Spec} A \otimes_k K$ is connected for any field extension K/k. It suffices to assume that K is algebraically closed (as $\operatorname{Spec} A \otimes_k \overline{K} \to \operatorname{Spec} A \otimes_k K$ is surjective). By choosing an embedding $\overline{k} \hookrightarrow K$ and considering the diagram

$$\begin{tikzcd}
\operatorname{Spec} A \otimes_k K \arrow[r] \arrow[d] & \operatorname{Spec} A \otimes_k \overline{k} \arrow[r, "\text{homeo.}", "\text{by Lem. 2.2}"'] \arrow[d] & \operatorname{Spec} A \arrow[d] \\
\operatorname{Spec} A \arrow[r] & \operatorname{Spec} \overline{k} \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

it suffices to assume k is algebraically closed.

If $\operatorname{Spec} A \otimes_k K$ is disconnected, then $A \otimes_k K$ contains an idempotent $e \neq 0, 1$ (by Exercise 3.3). By the tensor-finiteness trick, we may assume that A is a finitely generated algebra over k, and K is a finitely generated field extension. Write $K = FF(B)$ for an integral domain B of finite type over k. Then by the tensor-finiteness trick, by considering the finite number of denominators appearing in a representative of e as a sum of decomposable tensors, $e \in A \otimes_k B[1/b]$ for some nonzero $b \in B$, so $\operatorname{Spec} A \otimes_k B[1/b]$ is disconnected, say with disjoint opens U and V with $U \coprod V = \operatorname{Spec} A \otimes_k B[1/b]$.

Now $\phi : \operatorname{Spec} A \otimes_k B[1/b] \to \operatorname{Spec} B[1/b]$ is an open map (Proposition 1.1), so $\phi(U)$ and $\phi(V)$ are nonempty open sets. As $\operatorname{Spec} B[1/b]$ is connected, the intersection $\phi(U) \cap \phi(V)$ is a nonempty open set, which has a closed point p (with residue field k, as $k = \overline{k}$). But then $\phi^{-1}(p) \cong \operatorname{Spec} A$, and we have covered $\operatorname{Spec} A$ with two disjoint open sets, yielding a contradiction. □

3.5. *Corollary. — If k is separably closed, and Y is a connected k-scheme, then Y is geometrically connected.*

Proof. We wish to show that for any field extension K/k, Y_K is connected. By Proposition 3.4, $\operatorname{Spec} K$ is geometrically connected over k. Then apply Lemma 3.2 with $X = \operatorname{Spec} K$. □

4. Irreducibility

4.1. *Proposition. — Suppose k is separably closed, A is a k-algebra with $\operatorname{Spec} A$ irreducible, and K/k is a field extension. Then $\operatorname{Spec} A \otimes_k K$ is irreducible.*

Proof. We follow the philosophy of the proof of Proposition 3.4. As in the first paragraph of that proof, it suffices to assume that K and k are algebraically closed. If $A \otimes_k K$ is not irreducible, then we can find x and y with $V(x), V(y) \neq \operatorname{Spec} A \otimes_k K$ and $V(x) \cup V(y) = \operatorname{Spec} A \otimes_k K$. As in the second paragraph of the proof of Proposition 3.4, we may assume that A is a finitely generated algebra over k, and $K = FF(B)$ for an integral domain B of finite type over k, and $x, y \in A \otimes_k B[1/b]$ for some nonzero $b \in B$. Then $D(x)$ and $D(y)$ are nonempty open subsets of $\operatorname{Spec} A \otimes_k B[1/b]$, whose image in $\operatorname{Spec} B[1/b]$ are nonempty opens, and thus their intersection is nonempty and contains a closed point p. But then $\phi^{-1}(p) \cong \operatorname{Spec} A$, and we have covered $\operatorname{Spec} A$ with two proper closed sets (the restrictions of $V(x)$ and $V(y)$), yielding a contradiction. □

4.2. Exercise. Suppose k is separably closed, and A and B are k-algebras, both irreducible (with irreducible Spec, i.e. with one minimal prime). Show that $A \otimes_k B$ is irreducible too. (Hint: reduce to the case where A and B are finite type over k. Extend the proof of the previous proposition.)

4.3. Easy exercise. Show that a scheme X is irreducible if and only if there exists an open cover $X = \cup U_i$ with U_i irreducible for all i, and $U_i \cap U_j \neq \varnothing$ for all i, j.

4.4. *Proposition. — Suppose K/k is a field extension of a separably closed field and X_k is irreducible. Then X_K is irreducible.*

Proof. Take $X = \cup U_i$ irreducible as in Exercise 4.3. The base change of each U_i to K is irreducible by Proposition 4.1, and pairwise intersect. The result then follows from Exercise 4.3. □

5. Reducedness

We recall the following fact from field theory.

5.1. *Algebraic fact. — Suppose* E/F *is a finitely generated extension of a perfect field. Then it can be factored into a finite separable part and a purely transcendent part:* $E/F(t_1, \ldots, t_n)/F$.

5.2. *Proposition* [Stacks, 034N]. — *Suppose* B *is a geometrically reduced* k*-algbra, and* A *is a reduced k-algebra. Then* $A \otimes_k B$ *is reduced.*

Proof. Reduce to the case where A is finitely generated over k using the tensor-finiteness trick. (Suppose we have $x \in A \otimes_k B$ with $x^n = 0$. Then $x = \sum a_i \otimes b_i$. Let A' be the finitely generated subring of A generated by the a_i. Then $A' \otimes_k B$ is a subring of $A \otimes_k B$. Replace A by A'.) Then A is a subring of the product $\prod K_i$ of the function fields of its irreducible components (from our discussion on associated points). So it suffices to prove it for A a product of fields. Then it suffices to prove it when A is a field. But then we are done, by the definition of geometric reducedness. $\square$

5.3. *Propostion. — Suppose A is a reduced k-algebra. Then:*
(a) $A \otimes_k k(t)$ *is reduced.*
(b) If E/k *is a finite separable extension, then* $A \otimes_k E$ *is reduced.*

Proof. (a) Clearly $A \otimes k[t]$ is reduced, and localization preserves reducedness (as reducedess is stalk-local).

(b) Working inductively, we can assume E is generated by a single element, with minimal polynomial $p(t)$. By the tenor-finiteness trick, we can assume A is finitely generated over k. Then by the same trick as in the proof of Proposition 5.2, we can replace A by the product of its function fields of its components, and then we can assume A is a field. But then $A[t]/p(t)$ is reduced by the definition of separability of p. $\square$

5.4. *Lemma* [Stacks, 00I4 part 1]. — *Suppose* E/k *is a field extension of a perfect field, and* A *is a reduced* k*-algebra. Then* $A \otimes_k E$ *is reduced.*

Proof. By the tensor product finiteness trick, we may assume E is finitely generated over k. By Algebraic Lemma 5.1, we can factor E/k into extensions of the forms of Proposition 5.3 (a) and (b). We then apply Proposition 5.3. $\square$

5.5. *Proposition. — Suppose* E/k *is an extension of a perfect field, and* X *is a* k*-scheme. Then* X_E *is reduced.*

Proof. Reduce to the case where X is affine. Use Lemma 5.4. $\square$

5.6. *Corollary* [Stacks, 00I4 part 2]. — *Suppose* k *is perfect, and* A *and* B *are reduced* k*-algebras. The* $A \otimes_k B$ *is reduced.*

Proof. By Lemma 5.4, A is a geometrically reduced k-algebra. Then apply Lemma 5.2. $\square$

6. Varieties

Recall that I defined varieties to be finite type separated reduced schemes over k.

6.1. *Theorem.* — *(a) If* k *is perfect, the product of* k*-varieties (over* $\operatorname{Spec} k$*) is a* k*-variety.*
(b) If k *is algebraically closed, the product of irreducible* k*-varieties is an irreducible* k*-variety.*
(c) If k *is separably closed, the product of connected* k*-varieties is a connected* k*-variety.*

Proof. (a) The finite type and separated statements are done in the course notes (the first is easy). For reducedness, reduce to the affine case, then use Corollary 5.6. $\square$

(b) It only remains to show irreducibility. Reduce to the affine case using Exercise 4.3 (as in the proof of Proposition 4.4). Then use Proposition 4.2.

(c) This follows from Corollary 3.5. $\square$

References

[Stacks] The Stacks Project Authors, *Stacks Project*, `http://math.columbia.edu/algebraic_geometry/stacks-g`

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