

**MATH 216: FOUNDATIONS OF ALGEBRAIC  
GEOMETRY**

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## CHAPTER 1

### Introduction

*I can illustrate the ... approach with the ... image of a nut to be opened. The first analogy that came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better, and otherwise you left time pass. The shell becomes more flexible through weeks and months — when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado!*

*A different image came to me a few weeks ago. The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration ... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it ... yet finally it surrounds the resistant substance.*

— Grothendieck, *Récoltes et Semailles* p. 552-3, translation by Colin McLarty

### 1.1 Goals

These will hopefully eventually be complete notes intended to accompany a hard year-long class taught at Stanford in 2009-2010. In any class, choices must be made as to what the course is about, and who it is for — there is a finite amount of time, and any addition of material or explanation or philosophy requires a corresponding subtraction. So these notes are highly inappropriate for most people and most classes. Here are my goals. (I do not claim that these goals are achieved; but they motivate the choices made.)

These notes have a very particular audience in mind: Stanford Ph.D. students, postdocs and faculty in a variety of fields, who may want to use algebraic geometry in a sophisticated way. This includes algebraic and arithmetic geometers, but also topologists, number theorists, symplectic geometers, and others.

The notes deal purely with the algebraic side of the subject, and completely neglect analytic aspects.

They assume little prior background (see §1.2), and indeed most students have little prior background. Readers with less background will necessarily have to work harder. It would be great if the reader had seen varieties before, but many students haven't, and the course does not assume it — and similarly for category theory, homological algebra, more advanced commutative algebra, differential geometry, .... Surprisingly often, what we need can be developed quickly from scratch. The cost is that the course is much denser; the benefit is that more people can follow it; they don't reach a point where they get thrown. (On the other hand, people who already have some familiarity with algebraic geometry, but want to understand the foundations more completely should not be bored, and will focus on more subtle issues.)

The notes seek to cover everything that one should see in a first course in the subject, including theorems, proofs, and examples.

They seek to be complete, and not leave important results as black boxes pulled from other references.

There are lots of exercises. I've found that unless I have some problems I can think through, ideas don't get fixed in my mind. Some are trivial — that's okay, and even desirable. As few necessary ones as possible should be hard, but the reader should have the background to deal with them — they are not just an excuse to push material out of the text.

There are optional (starred  $\star$ ) sections of topics worth knowing on a second or third (but not first) reading. You should not read double-starred sections ( $\star\star$ ) unless you really really want to, but you should be aware of their existence.

The notes are intended to be readable, even if not easy reading.

In short, after a year of hard work, students should have a broad familiarity with the foundations of the subject, and be ready to attend seminars, and learn more advanced material. They should not just have a vague intuitive understanding of the ideas of the subject; they should know interesting examples, know why they are interesting, and be able to prove interesting facts about them.

I have greatly enjoyed thinking through these notes, and teaching the corresponding classes, in a way I did not expect. I have had the chance to think through the structure of algebraic geometry from scratch, not blindly accepting the choices made by others. (Why do we need this notion? Aha, this forces us to consider this other notion earlier, and now I see why this third notion is so relevant...) I have repeatedly realized that ideas developed in Paris in the 1960's are simpler than I initially believed, once they are suitably digested.

**1.1.1. Implications.** We will work with as much generality as we need for most readers, and no more. In particular, we try to have hypotheses that are as general as possible without making proofs harder. The right hypotheses can make a proof easier, not harder, because one can remember how they get used. As an inflammatory example, the notion of quasiseparated comes up early and often. The cost is that one extra word has to be remembered, on top of an overwhelming number of other words. But once that is done, it is not hard to remember that essentially every scheme anyone cares about is quasiseparated. Furthermore, whenever the hypotheses "quasicompact and quasiseparated" turn up, the reader will likely immediately see a key idea of the proof.

Similarly, there is no need to work over an algebraically closed field, or even a field. Geometers needn't be afraid of arithmetic examples or of algebraic examples; a central insight of algebraic geometry is that the same formalism applies without change.

**1.1.2. Costs.** Choosing these priorities requires that others be shortchanged, and it is best to be up front about these. Because of our goal is to be comprehensive, and to understand everything one should know after a first course, it will necessarily take longer to get to interesting sample applications. You may be misled into thinking that one has to work this hard to get to these applications — it is not true!

## 1.2 Background and conventions

All rings are assumed to be commutative unless explicitly stated otherwise. All rings are assumed to contain a unit, denoted 1. Maps of rings must send 1 to 1. We don't require that  $0 \neq 1$ ; in other words, the "0-ring" (with one element) is a ring. (There is a ring map from any ring to the 0-ring; the 0-ring only maps to itself. The 0-ring is the final object in the category of rings.) We accept the axiom of choice. In particular, any proper ideal in a ring is contained in a maximal ideal. (The axiom of choice also arises in the argument that the category of  $A$ -modules has enough injectives, see Exercise 23.2.E.)

The reader should be familiar with some basic notions in commutative ring theory, in particular the notion of ideals (including prime and maximal ideals) and localization. For example, the reader should be able to show that if  $S$  is a multiplicative set of a ring  $A$ , then the primes of  $S^{-1}A$  are in natural bijection with those primes of  $A$  not meeting  $S$  (§4.2.6). The notion of tensor products and exact sequences of  $A$ -modules will be important. We will use the notation  $(A, \mathfrak{m})$  or  $(A, \mathfrak{m}, k)$  for local rings —  $A$  is the ring,  $\mathfrak{m}$  its maximal ideal, and  $k = A/\mathfrak{m}$  its residue field. We will use (in Proposition 14.6.4) the structure theorem for finitely generated modules over a principal ideal domain  $A$ : any such module can be written as the direct sum of principal modules  $A/(a)$ .

We will not concern ourselves with subtle foundational issues (set-theoretic issues involving universes, etc.). It is true that some people should be careful about these issues. But is that really how you want to spend your life?

**1.2.1. Further background.** It may be helpful to have books on other subjects handy that you can dip into for specific facts, rather than reading them in advance. In commutative algebra, Eisenbud [E] is good for this. Other popular choices are Atiyah-Macdonald [AM] and Matsumura [M-CRT]. For homological algebra, Weibel [W] is simultaneously detailed and readable.

Background from other parts of mathematics (topology, geometry, complex analysis) will of course be helpful for developing intuition.

Finally, it may help to keep the following quote in mind.

*Algebraic geometry seems to have acquired the reputation of being esoteric, exclusive, and very abstract, with adherents who are secretly plotting to take over all the rest of mathematics. In one respect this last point is accurate.*

— David Mumford