MATH 120 PRACTICE FINAL

Give complete arguments. The problems are not in order of increasing difficulty. All rings are assumed to be commutative with 1. If you have any questions about the problems, or what you are allowed to use, please ask. Good luck!

1. (a) Show that if n is not prime, then $\mathbb{Z}/n\mathbb{Z}$ is not a field.

(b) Prove that a group of order 312 has a normal Sylow p-subgroup for some prime p dividing its order.

2. Let H be a subgroup of the additive group of rational numbers with the property that $1/x \in H$ for every nonzero element x of H. Prove that $H = 0$ or $\mathbb{Q}$.

3. Suppose G is an infinite group, and H and K are finite subgroups whose orders are relatively prime. Show that $H \cap K = \{e\}$.

4. Suppose G is a subgroup of S_n. Let H be the subset of G of even elements. Show that H is a normal subgroup of G of index 1 or 2.

5. Suppose G is a finite abelian group, with at most n elements of order n, for all n. Show that G is cyclic. (Hint: use the fundamental theorem of finitely generated abelian groups.)

6. Show that there is a nonabelian group of order 21. (Hint: construct an appropriate semidirect product.)

7. Show that any subgroup of a nilpotent group is nilpotent.

8. Prove that if R is an integral domain, then the ring of formal power series $R[[x]]$ is also an integral domain. (Recall that $R[[x]]$ consisted of "infinite polynomials", of the form

$$r_0 + r_1 x + r_2 x^2 + r_3 x^3 + \cdots$$

where $r_i \in R$.)

9. Suppose R is a unique factorization domain, and D is a multiplicative subset, not containing zero. Show that $D^{-1}R$ (the ring of fractions with denominators in D) is also a unique factorization domain. (Hence for example those rational numbers with only a power of 2 in the denominator form a unique factorization domain.) Hint: show that some irreducibles in R stay irreducible in $D^{-1}R$, and others become units.

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