Dear 120-ites,

Here are your midterms back, 2 days earlier than expected – grading isn't that hard a job after all! You guys gave lots of nice answers, it was tough to choose just one per question to feature here.

1a

Marc Rasi

Index of stabilizer = order of orbit

Since action is transitive, order of orbit is 120.

$\boxed{\text{So Index of Stabilizer} = 120}$

1b

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$$|A_4| = \frac{|S_4|}{2} = \frac{4!}{2} = 12$$

since this question doesn't ask for a proof, you shouldn't say any more than Marc's answer. less than this would be fine too.

Ted Westling

2. (a) $8 = |Z(G)| = 8$ since $\mathbb{Z}/8\mathbb{Z}$ is abelian.

(b) 6 subgroups (4 proper subgroups) $G, \{e\}, \langle r \rangle\ \langle s \rangle\ \langle sr \rangle, \langle sr^2 \rangle$

For b, don't forget that sr and sr^2 are also elements of order 2!

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Thomas Davids
Problem 3

This group can be entirely generated by 5.

Proof. Any element in the subgroup generated by 10 and 25 can be written as $g \equiv 10 \cdot a + 25 \cdot b$, (mod 100) for some $a, b \in \mathbb{Z}$.

So $g \equiv 5 \cdot 2a + 5 \cdot 5b \equiv 5(2a + 5b)$, and since $2a + 5b \in \mathbb{Z}$, this proves that $\langle 10, 25 \rangle \subseteq \langle 5 \rangle$.

But any element generated by 5 can be expressed as $5 \cdot c \pmod{100} \equiv 25 \cdot c - 10 \cdot 2c$, and since c and $2c \in \mathbb{Z}$, this proves that $\langle 5 \rangle \subseteq \langle 10, 25 \rangle$, which proves equality.

Any group generated by a single element is by definition cyclic, so: this this group is cyclic.

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Observe there are two parts to Thomas's answer:

- everything in <10,25> is a multiple of 5
- we can put 5 in the form $10a+25b$

#4

Sally Hudson

Let $\varphi: A_5 \to \mathbb{Z}_n$ be a homomorphism from A_5 to a cyclic group of order n. The kernel of φ must be normal in A_5. Since A_5 is simple, the only normal subgroups of A_5 are $\{e\}$ and A_5.

Case 1: $\ker \varphi = A_5$

By the 1st Isomorphism Theorem, $A_5 / \ker \varphi \cong \operatorname{im} \varphi$. By Langrange's Theorem, $|A_5 / \ker \varphi| = \dfrac{|A_5|}{|\ker \varphi|} = \dfrac{|A_5|}{|A_5|} = 1$

Thus, $|\operatorname{im} \varphi| = 1$. Since the image of φ must be a subgroup of $\mathbb{Z}_n$, it follows that $\operatorname{im} \varphi = \{e\}$, as desired

Case 2: $\ker \varphi = \{e\}$

We will show that this case is not possible. If $\ker \varphi = \{e\}$, then $A_5 \cong \operatorname{im} \varphi$. by 1st Isomorphism Theorem Because $\operatorname{im} \varphi$ is a subgroup of the cyclic group, and cyclic groups are abelian, we must have that $\operatorname{im} \varphi$ is abelian and therefore A_5 is abelian. But A_5 is not abelian, so $\ker \varphi \neq \{e\}$.

Of course, a good answer need not be as long as this, if you cover all the important points clearly.

Q5, solution by Jeffrey Chen

The three *ed points are the important things here.

(a)

The 3rd Sylow theorem tells us that if $|G| = p^a m$, $p \nmid m$, then the number n_p of Sylow p-groups satisfies $n_p \mid m$ and $n_p \equiv 1 \pmod p$.

Thus, $n_2 \mid 7$ and $n_2 \equiv 1 \pmod 2$, so G has either 1 or 7 Sylow 2-groups.

Also, $n_7 \mid 8$ and $n_7 \equiv 1\ (7)$, so G has either 1 or 8 Sylow 7-subgroups.

(b) Suppose, by way of contradiction, that G were simple. Then it cannot have only 1 Sylow 2-subgroup, as all conjugates of such a group would be itself and it would therefore be normal. Similarly, G cannot have only 1 Sylow 7-group. Hence, G has 7 Sylow 2-subgroups and 8 Sylow 7-subgroups. Now every element of a Sylow 7-subgroup (except the identity) has order 7 and any one of them will generate the entire subgroup, so all the Sylow 7-groups ~~are~~ intersect trivially ~~disjoint~~* and thus we have $8 \cdot (7-1) = 48$ elements of order 7.* Take P to be a Sylow 2-subgroup, so we know $|P| = 8$. Now P has no elements of order 7,* since $7 \nmid |P|$. Moreover, if P' is another Sylow 2-subgroup, it must have at least one element not in P, otherwise P and P' would be identical. Hence, we have at least 9 elements that do not have order 7. However, this gives at least $48 + 9 = 57$ elements, which is a contradiction since G has only 56 elements.* Thus, G is not simple. ∎

Question 6

This is hard; no one got it.

Let $A = \{a_1, a_2, \cdots a_n\}$

Recall the stabiliser $G_{a_i} = \{\sigma \in G \mid \sigma(a_i) = a_i\} \subseteq G$

$\therefore$ elements of G which fix an object $= G_{a_1} \cup G_{a_2} \cup \cdots \cup G_{a_n}$.

G is transitive $\therefore$ all points of A are in the same orbit, which has size n.

$\Rightarrow$ all stabilisers have size $|G|/n$.

The identity $\in$ every stabiliser

so $G_{a_1} \cup G_{a_2} \cup \cdots \cup G_{a_n}$ has size at most $1 + n(|G|/n - 1) = |G| - (n-1) < G$ if $n > 1$

$\therefore$ G must contain permutations which fix no points.

If you have any questions, please drop by during office hours or email me: amypang@stanford.edu.

— Amy