

Dear 120-ites,

Here are your midterms back, 2 days earlier than expected - grading isn't that hard a job after all! You guys gave lots of nice answers, it was tough to choose just one per question to feature here.

(7a)

Marc Rasi

Index of stabilizer = order of orbit

Since action is transitive, order of orbit is

120.

So Index of stabilizer = 120

(1b)

$$|A_4| = \frac{|S_4|}{2} = \frac{4!}{2} = 12$$

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since this question doesn't ask for a proof, you shouldn't say any more than Marc's answer. less than this would be fine too.

Ted Westling

2. (a)  $8 = |Z(G)| = 8$  since  $\mathbb{Z}/8\mathbb{Z}$  is abelian.

(b) 6 subgroups (4 proper subgroups)  $G, \{e\}, \langle r \rangle, \langle s \rangle, \langle sr \rangle, \langle sr^2 \rangle$

For b, don't forget that  $sr$  and  $sr^2$  are also elements of order 2!

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Thomas David's  
Problem 3

This group can be entirely generated by 5.

Proof. Any element in the subgroup generated by 10 and 25 can be written as  $g \equiv 10 \cdot a + 25 \cdot b \pmod{100}$  for some  $a, b \in \mathbb{Z}$ .

So  $g \equiv 5 \cdot 2a + 5 \cdot 5b \equiv 5(2a + 5b)$ , and since  $2a + 5b \in \mathbb{Z}$ , this proves that  $\langle 10, 25 \rangle \subseteq \langle 5 \rangle$ .

But any element generated by 5 can be expressed as  $5 \cdot c \pmod{100} \equiv 25 \cdot c - 10 \cdot 2c$ , and since  $c$  and  $2c \in \mathbb{Z}$ , this proves that  $\langle 5 \rangle \subseteq \langle 10, 25 \rangle$ , which proves equality.

Any group generated by a single element is by definition cyclic, so this group is cyclic.

Observe there are two parts to Thomas's answer:

- everything in  $\langle 10, 25 \rangle$  is a multiple of 5
- we can put 5 in the form  $10a + 25b$

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#4

Sally Hudson

Let  $\varphi: A_5 \rightarrow \mathbb{Z}_n$  be a homomorphism from  $A_5$  to a cyclic group of order  $n$ . The kernel of  $\varphi$  must be normal in  $A_5$ . Since  $A_5$  is simple, the only normal subgroups of  $A_5$  are  $\{e\}$  and  $A_5$ .

Case 1:  $\ker \varphi = A_5$

By the 1st Isomorphism Theorem,  $A_5 / \ker \varphi \cong \text{im } \varphi$ .

By Lagrange's Theorem,  $|A_5 / \ker \varphi| = \frac{|A_5|}{|\ker \varphi|} = \frac{|A_5|}{|A_5|} = 1$

Thus,  $|\text{im } \varphi| = 1$ . Since the image of  $\varphi$  must be a subgroup of  $\mathbb{Z}_n$ , it follows that  $\text{im } \varphi = \{e\}$ , as desired.

Case 2:  $\ker \varphi = \{e\}$

We will show that this case is not possible.

If  $\ker \varphi = \{e\}$ , then  $A_5 \cong \text{im } \varphi$  by 1st Isomorphism Theorem. Because  $\text{im } \varphi$  is a subgroup of the cyclic group, and cyclic groups are abelian, we must have that  $\text{im } \varphi$  is abelian and therefore  $A_5$  is abelian. But  $A_5$  is not abelian, so  $\ker \varphi \neq \{e\}$ .

Of course, a good answer need not be as long as this, if you cover all the important points clearly.

Q5, solution by Jeffrey Chen

The three \*ed points are the important things here.

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The 3rd Sylow theorem tells us that if  $|G| = p^a m$ ,  $p \nmid m$ , then the number  $n_p$  of Sylow  $p$ -groups satisfies  $n_p \mid m$  and  $n_p \equiv 1 \pmod{p}$ .

Thus,  $n_2 \mid 7$  and  $n_2 \equiv 1 \pmod{2}$ , so  $G$  has either 1 or 7 Sylow 2-subgroups.

Also,  $n_7 \mid 8$  and  $n_7 \equiv 1 \pmod{7}$ , so  $G$  has either 1 or 8 Sylow 7-subgroups.

- (b) Suppose, by way of contradiction, that  $G$  were simple. Then it cannot have only 1 Sylow 2-subgroup, as all conjugates of such a group would be itself and it would therefore be normal. Similarly,  $G$  cannot have only 1 Sylow 7-subgroup. Hence,  $G$  has 7 Sylow 2-subgroups and 8 Sylow 7-subgroups. Now every element of a Sylow 7-subgroup (except the identity) has order 7 and any one of them will generate the entire subgroup, so all the Sylow 7-groups ~~are~~ intersect ~~trivially~~ <sup>disjoint\*</sup> and thus we have  $8 \cdot (7-1) = 48$  elements of order 7. Take  $P$  to be a Sylow 2-subgroup, so we know  $|P| = 8$ . Now  $P$  has  $n$  elements of order 7, since  $7 \nmid |P|$ . Moreover, if  $P'$  is another Sylow 2-subgroup, it must have at least one element not in  $P$ , otherwise  $P$  and  $P'$  would be identical. Hence, we have at least 9 elements that do not have order 7. However, this gives at least  $48 + 9 = 57$  elements, which is a contradiction since  $G$  has only 56 elements\*. Thus,  $G$  is not simple.  $\square$

## Question 6

This is hard; no one got it.

Let  $A = \{a_1, a_2, \dots, a_n\}$

Recall the stabiliser  $G_{a_i} = \{\sigma \in G \mid \sigma(a_i) = a_i\} \subseteq G$

$\therefore$  elements of  $G$  which fix an object  $= G_{a_1} \cup G_{a_2} \cup \dots \cup G_{a_n}$ .

$G$  is transitive  $\therefore$  all points of  $A$  are in the same orbit, which has size  $n$ .

$\Rightarrow$  all stabilisers have size  $|G|/n$ .

The identity  $\in$  every stabiliser

so  $G_{a_1} \cup G_{a_2} \cup \dots \cup G_{a_n}$  has size at most  $1 + n(|G|/n - 1) = |G| - (n-1) < |G|$  if  $n > 1$

$\therefore G$  must contain permutations which fix no points.

If you have any questions, please drop by during office hours or email me:  
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— Amy