MATH 120 MIDTERM

Write your name at the top of each page. Give complete proofs except for problem 1, where answers will suffice. Each problem is worth the same. If you would like to know how you did before the drop date (Sunday), please send me an e-mail, and I'll respond on Saturday.

1.
(a) Is the set of rational numbers in lowest terms whose denominators are even, along with zero, a subgroup of the (additive group of) rational numbers?
(b) How many elements of D_8 have order exactly 2?
(c) Must every subgroup of an abelian group be normal?

2. Show that if H and K are normal subgroups of G, then $H \cap K$ is normal in G too.

3. Let M and N be normal subgroups of G such that $G = MN$. Prove that

$$G/(M \cap N) \cong (G/M) \times (G/N).$$

4. Show that $GL_2(\mathbb{F}_2)$ is isomorphic to S_3.

5. Let G be any group. (a) Prove that the map $G \to G$ defined by $g \mapsto g^2$ is a homomorphism if and only if G is abelian. (b) If G is abelian and finite show that this map is an isomorphism if and only if G has odd order.

6. Let G be a group of order pq, where $p > q$ are primes.
(a) Show that there is one subgroup of order p in G. (Hint: if A and A' were two such, what can you say about $|AA'|$?)
(b) Suppose $a \in G$ has order p. Show that $A = \langle a \rangle$ (the subgroup generated by a) is normal in G.
(c) Show that if $x \in G$, then $x^{-1}ax = a^i$ for some $0 < i < p$ (depending on x).
(d) *(harder)* Show that if q is not a factor of $p - 1$, then G is cyclic.

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