FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 7

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This set is due at noon on Friday November 16. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers class 13.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions, where each "-" problem is worth half a solution, each "+" problem is worth one-and-a-half, and each "++" problem is worth two. *You are allowed to hand in up to two problems from previous sets that you have not done.* If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1+. *(a useful criterion for when ideals in affine open sets define a closed subscheme)* It will be convenient to define certain closed subschemes of Y by defining on any affine subset $\operatorname{Spec} B$ of Y an ideal $I_B \subset B$. Show that these $\operatorname{Spec} B/I_B \hookrightarrow \operatorname{Spec} B$ glue together to form a closed subscheme precisely if for each affine open subset $\operatorname{Spec} B \hookrightarrow Y$ and each $f \in B$, $I_{(B_f)} = (I_B)_f$.

You might hope that closed subschemes correspond to ideal sheaves of $\mathcal{O}_Y$. Sadly not every ideal sheaf arises in this way. Here is an example.

2. Let $X = \operatorname{Spec} k[x]_{(x)}$, the germ of the affine line at the origin, which has two points, the closed point and the generic point η. Define $\mathcal{I}(X) = \{0\} \subset \mathcal{O}_X(X) = k[x]_{(x)}$, and $\mathcal{I}(\eta) = k(x) = \mathcal{O}_X(\eta)$. Show that this sheaf of ideals does not correspond to a closed subscheme.

3. (a) Show that $wz = xy, x^2 = wy, y^2 = xz$ describes an irreducible curve in $\mathbb{P}^3_k$. This curve is called the *twisted cubic*. The twisted cubic is a good non-trivial example of many things, so it you should make friends with it as soon as possible. (b) Show that the twisted cubic is isomorphic to $\mathbb{P}^1_k$.

4. The usual definition of a closed immersion is a morphism $f : X \to Y$ such that f induces a homeomorphism of the underlying topological space of Y onto a closed subset of the topological space of X, and the induced map $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves on X is surjective. Show that this definition agrees with the one given above. (To show that our definition involving surjectivity on the level of affine open sets implies this definition, you can use the fact that surjectivity of a morphism of sheaves can be checked on a base, which you can verify yourself.)

Date: Friday, November 9, 2007.

5. Suppose X is an affine scheme, and Y is a closed subscheme locally cut out by one equation (e.g. if Y is an effective Cartier divisor). Show that $X - Y$ is affine. (This is clear if Y is globally cut out by one equation f; then if $X = \operatorname{Spec} A$ then $Y = \operatorname{Spec} A_f$. However, Y is not always of this form.)

6-. If X is reduced, show that the scheme-theoretic image of $f : X \to Y$ is also reduced.

7. If $f : X \to Y$ is a morphism of locally Noetherian schemes, show that the associated points of the image subscheme are a subset of the image of the associated points of X.

8. Suppose X is a Noetherian scheme. Show that a subset of X is constructible if and only if it is the finite disjoint union of locally closed subsets.

9. If $X \to Y$ is quasicompact and quasiseparated (e.g. if X is Noetherian) or if X is reduced, show that the following three notions are the same.

(a) V is an open subscheme of X intersect a closed subscheme of X
(b) V is an open subscheme of a closed subscheme of X
(c) V is a closed subscheme of an open subscheme of X.

(Hint: it will be helpful to note that the scheme-theoretic image may be computed on each open subset of the base.)

10. If $f : X \to Y$ is a locally closed immersion into a locally Noetherian scheme (so X is also locally Noetherian), then the associated points of the scheme-theoretic image are (naturally in bijection with) the associated points of X. (Hint: Exercise .) Informally, we get no non-reduced structure on the scheme-theoretic closure not "forced by" that on X.

11. *(the notions "locally of finite type" and "finite type" are affine-local on the target)* Show that a morphism $f : X \to Y$ is locally of finite type if there is a cover of Y by open affine sets $\operatorname{Spec} B_i$ such that $f^{-1}(\operatorname{Spec} B_i)$ is locally of finite type over B_i.

12. Show that a morphism $f : X \to Y$ is locally of finite type if for *every* affine open subsets $\operatorname{Spec} A \subset X$, $\operatorname{Spec} B \subset Y$, with $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, A is a finitely generated B-algebra. (Hint: use the affine communication lemma on $f^{-1}(\operatorname{Spec} B)$.)

13-. Show that finite morphisms are of finite type. Hence closed immersions are of finite type.

14+. *(not hard, but important)*

(a) Show that an open immersion is locally of finite type. Show that an open immersion into a locally Noetherian scheme is of finite type. More generally, show that a quasicompact open immersion is of finite type.
(b) Show that the composition of two morphisms of locally finite type is locally of finite type. (Hence as quasicompact morphisms also compose, the composition of two morphisms of finite type is also of finite type.)

(c) Suppose we have morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, with f quasicompact, and $g \circ f$ of finite type. Show that f is finite type.
(d) Suppose $f : X \to Y$ is finite type, and Y is Noetherian. Show that X is also Noetherian.

The following are double-plus problems because I'd like to see people try them.

15++. Show that the notion of "locally finite presentation" is affine-local.

16++. A scheme is *quasiseparated* if the intersection of two affine open sets is the finite union of affine schemes. Show that this notion is affine-local.

17++. A morphism is *quasiseparated* if the preimage of every affine scheme is a quasiseparated scheme. Show that this notion is affine-local on the target.

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