FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 6

RAVI VAKIL

This set is due at noon on Friday November 9. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 11 and 12.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in nine solutions, where each "-" problem is worth half a solution and each "+" problem is worth one-and-a-half. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1. *(associated primes behave well with respect to localization)* Show that if A is a Noetherian ring, and S is a multiplicative subset (so there is an inclusion-preserving correspondence between the primes of $S^{-1}A$ and those primes of A not meeting S), then the associated primes of $S^{-1}A$ are just the associated primes of A not meeting S.

2. (a) Show that the minimal primes of A are associated primes. We have now proved important fact **(1)**. (Hint: suppose $\mathfrak{p} \supset \cap_{i=1}^n \mathfrak{q}_i$. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} \supset \sqrt{\cap_{i=1}^n \mathfrak{q}_i} = \cap_{i=1}^n \sqrt{\mathfrak{q}_i} = \cap_{i=1}^n \mathfrak{p}_i$, so by a previous exercise, $\mathfrak{p} \supset \mathfrak{p}_i$ for some i. If $\mathfrak{p}$ is minimal, then as $\mathfrak{p} \supset \mathfrak{p}_i \subset (0)$, we must have $\mathfrak{p} = \mathfrak{p}_i$.)
(b) Show that there can be other associated primes that are not minimal. (Hint: we've seen an example...) Your argument will show more generally that the minimal primes of I are associated primes of I.

3. Show that if A is reduced, then the only associated primes are the minimal primes. (This establishes **(2)**.)

4. Show that

$$Z = \cup_{x \neq 0}(0 : x) \subseteq \cup_{x \neq 0}\sqrt{(0 : x)} \subseteq Z.$$

5-. *(Rabinoff's Theorem)* Here is an interesting variation on **(4)**: show that $a \in A$ is nilpotent if and only if it vanishes at the associated points of $\operatorname{Spec} A$. Algebraically: we know that the nilpotents are the intersection of *all* prime ideals; now show that in the Noetherian case, the nilpotents are in fact the intersection of the (finite number of) associated prime ideals.

6-. Prove fact **(3)**.

Date: Thursday, November 1, 2007. Updated November 9.

7-. Let $\vee\vee$: **f.d. Vec**$_k$ $\to$ **f.d. Vec**$_k$ be the double dual functor from the category of vector spaces over k to itself. Show that $\vee\vee$ is naturally isomorphic to the identity. (Without the finite-dimensional hypothesis, we only get a natural transformation of functors from id to $\vee\vee$.)

8-. Show that $\mathcal{V} \to$ **f.d.Vec**$_k$ gives an equivalence of categories, by describing an "inverse" functor. (You'll need the axiom of choice, as you'll simultaneously choose bases for each vector space in **f.d.Vec**$_k$!)

9. Assuming that morphisms of schemes are defined so that Motivation (a) holds, show that the category of rings and the opposite category of affine schemes are equivalent.

10. *(morphisms of ringed spaces glue)* Suppose $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed spaces, $X = \cup_i U_i$ is an open cover of X, and we have morphisms of ringed spaces $f_i : U_i \to Y$ that "agree on the overlaps", i.e. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Show that there is a unique morphism of ringed spaces $f : X \to Y$ such that $f|_{U_i} = f_i$. (An earlier exercise essentially showed this for topological spaces.)

11+. Given a morphism of ringed spaces $f : X \to Y$ with $f(p) = q$, show that there is a map of stalks $(\mathcal{O}_Y)_q \to (\mathcal{O}_X)_p$.

12++. Suppose $f^{\#} : B \to A$ is a morphism of rings. Define a morphism of ringed spaces $f : \operatorname{Spec} A \to \operatorname{Spec} B$ as follows. The map of topological spaces was given earlier. To describe a morphism of sheaves $\mathcal{O}_B \to f_*\mathcal{O}_A$ on $\operatorname{Spec} B$, it suffices to describe a morphism of sheaves on the distinguished base of $\operatorname{Spec} B$. On $D(g) \subset \operatorname{Spec} B$, we define

$$\mathcal{O}_B(D(g)) \to \mathcal{O}_A(f^{-1}D(g)) = \mathcal{O}_A(D(f^{\#}g))$$

by $B_g \to A_{f^{\#}g}$. Verify that this makes sense (e.g. is independent of g), and that this describes a morphism of sheaves on the distinguished base. (This is the third in a series of exercises. We showed that a morphism of rings induces a map of sets first, a map of topological spaces later, and now a map of ringed spaces here.)

13-. Recall that $\operatorname{Spec} k[x]_{(x)}$ has two points, corresponding to (0) and (x), where the second point is closed, and the first is not. Consider the map *of ringed spaces* $\operatorname{Spec} k(x) \to \operatorname{Spec} k[x]_{(x)}$ sending the point of $\operatorname{Spec} k(x)$ to $[(x)]$, and the pullback map $f^{\#}\mathcal{O}_{\operatorname{Spec} k(x)} \to \mathcal{O}_{\operatorname{Spec} k[x]_{(x)}}$ is induced by $k \hookrightarrow k(x)$. Show that this map of ringed spaces is not of the form described in Key Exercise .

14. Show that morphisms of locally ringed spaces glue

15+. (a) Show that $\operatorname{Spec} A$ is a locally ringed space. (b) The morphism of ringed spaces $f : \operatorname{Spec} A \to \operatorname{Spec} B$ defined by a ring morphism $f^{\#} : B \to A$ is a morphism of locally ringed spaces.

16+. Show that a morphism of schemes $f : X \to Y$ is a morphism of ringed spaces that looks locally like morphisms of affines. Precisely, if $\operatorname{Spec} A$ is an affine open subset of X and $\operatorname{Spec} B$ is an affine open subset of Y, and $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, then the induced morphism of ringed spaces is a morphism of affine schemes. Show that it suffices to check on a set $(\operatorname{Spec} A_i, \operatorname{Spec} B_i)$ where the $\operatorname{Spec} A_i$ form an open cover X.

17+. (This exercise will give you some practice with understanding morphisms of schemes by cutting up into affine open sets.) Make sense of the following sentence: "$\mathbb{A}^{n+1} \setminus \{\vec{0}\} \to \mathbb{P}^n$ given by

$$(x_0, x_1, \ldots, x_{n+1}) \mapsto [x_0; x_1; \ldots; x_n]$$

is a morphism of schemes." Caution: you can't just say where points go; you have to say where functions go. So you'll have to divide these up into affines, and describe the maps, and check that they glue.

18+. Show that morphisms $X \to \operatorname{Spec} A$ are in natural bijection with ring morphisms $A \to \Gamma(X, \mathcal{O}_X)$. (Hint: Show that this is true when X is affine. Use the fact that morphisms glue.)

19-. Show that $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes. In other words, if X is any scheme, there exists a unique morphism to $\operatorname{Spec} \mathbb{Z}$. (Hence the category of schemes is isomorphic to the category of $\mathbb{Z}$-schemes.)

20-. Show that morphisms $X \to \operatorname{Spec} \mathbb{Z}[t]$ correspond to global sections of the structure sheaf.

21-. Show that global sections of $\mathcal{O}_X^*$ correspond naturally to maps to $\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$. ($\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ is a *group scheme*.)

22. Suppose $i : U \to Z$ is an open immersion, and $f : Y \to Z$ is any morphism. Show that $U \times_Z Y$ exists. (Hint: I'll even tell you what it is: $(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)})$.)

23-. Show that open immersions are monomorphisms.

24. Show that a morphism $f : X \to Y$ is quasicompact if there is cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasicompact. (Hint: easy application of the affine communication lemma!)

25-. Show that the composition of two quasicompact morphisms is quasicompact.

26. *(the notions "locally of finite type" and "finite type" are affine-local on the target)* Show that a morphism $f : X \to Y$ is locally of finite type if there is a cover of Y by open affine sets $\operatorname{Spec} B_i$ such that $f^{-1}(\operatorname{Spec} B_i)$ is locally of finite type over B_i.

27. Show that a morphism $f : X \to Y$ is locally of finite type if for *every* affine open subsets $\operatorname{Spec} A \subset X$, $\operatorname{Spec} B \subset Y$, with $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, A is a finitely generated B-algebra. (Hint: use the affine communication lemma on $f^{-1}(\operatorname{Spec} B)$.)

28+. *(not hard, but important —)*

(a) Show that a closed immersion is a morphism of finite type.
(b) Show that an open immersion is locally of finite type. Show that an open immersion into a locally Noetherian scheme is of finite type. More generally, show that a quasicompact open immersion is of finite type.

(c) Show that the composition of two morphisms of locally finite type is locally of finite type. (Hence as quasicompact morphisms also compose, the composition of two morphisms of finite type is also of finite type.)

(d) Suppose we have morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, with f quasicompact, and $g \circ f$ of finite type. Show that f is finite type.

(e) Suppose $f : X \to Y$ is finite type, and Y is Noetherian. Show that X is also Noetherian.

29. *(the property of finiteness is affine-local on the target)* Show that a morphism $f : X \to Y$ is finite if there is a cover of Y by open affine sets $\operatorname{Spec} A$ such that $f^{-1}(\operatorname{Spec} A)$ is the spectrum of a finite A-algebra.

30-. Show that the composition of two finite morphisms is also finite.

31+. Show that finite morphisms are *closed*, i.e. the image of any closed subset is closed. (Hint: going-up theorem.)

32. (a) Show that if a morphism is finite then it is quasifinite. (b) Show that the converse is not true. (Hint: $\mathbb{A}^1 - \{0\} \to \mathbb{A}^1$.)

33. Show that the property of being a closed immersion is affine-local on the target.

34. In analogy with closed subsets, define the notion of a finite union of closed subschemes of X, and an arbitrary intersection of closed subschemes. Show that the underlying set of a finite union of closed subschemes is the finite union of the underlying sets, and similarly for arbitrary intersections.

35-. Show that closed immersions are finite morphisms.

E-mail address: `vakil@math.stanford.edu`