FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 2

RAVI VAKIL

This set is due at noon on Friday October 12. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 3 and 4.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in ten solutions, where each "-" problem is worth half a solution and each "+" problem is worth one-and-a-half. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 3.

1. Suppose

$$0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0$$

is a complex of k-vector spaces (often called $A^\bullet$ for short). Show that $\sum(-1)^i \dim A^i = \sum(-1)^i h^i(A^\bullet)$. (Recall that $h^i(A^\bullet) = \dim \ker(d^i)/\operatorname{im}(d^{i-1})$.) In particular, if $A^\bullet$ is exact, then $\sum(-1)^i \dim A^i = 0$. (If you haven't dealt much with cohomology, this will give you some practice.)

2. *(important)* Suppose $\mathcal{C}$ is an abelian category. Define the category $\mathbf{Com}_{\mathcal{C}}$ as follows. The objects are infinite complexes

$$A^\bullet : \qquad \cdots \longrightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \cdots$$

in $\mathcal{C}$, and the morphisms $A^\bullet \to B^\bullet$ are commuting diagrams

$$\begin{array}{ccccccccc} A^\bullet : & \cdots \longrightarrow & A^{i-1} & \xrightarrow{f^{i-1}} & A^i & \xrightarrow{f^i} & A^{i+1} & \xrightarrow{f^{i+1}} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ B^\bullet : & \cdots \longrightarrow & B^{i-1} & \xrightarrow{f^{i-1}} & B^i & \xrightarrow{f^i} & B^{i+1} & \xrightarrow{f^{i+1}} & \cdots \end{array}$$

Date: Wednesday, October 3, 2007.

Show that $\mathbf{Com}_{\mathcal{C}}$ is an abelian category. Show that a short exact sequence of complexes

$$\begin{array}{ccccccccccc}
0: & & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
A^\bullet: & & \cdots & \longrightarrow & A^{i-1} & \xrightarrow{f^{i-1}} & A^i & \xrightarrow{f^i} & A^{i+1} & \xrightarrow{f^{i+1}} & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
B^\bullet: & & \cdots & \longrightarrow & B^{i-1} & \xrightarrow{g^{i-1}} & B^i & \xrightarrow{g^i} & B^{i+1} & \xrightarrow{g^{i+1}} & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
C^\bullet: & & \cdots & \longrightarrow & C^{i-1} & \xrightarrow{h^{i-1}} & C^i & \xrightarrow{h^i} & C^{i+1} & \xrightarrow{h^{i+1}} & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
0: & & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

induces a *long exact sequence in cohomology*

$$\begin{array}{l}
\cdots \longrightarrow H^{i-1}(C^\bullet) \longrightarrow \\
H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(C^\bullet) \longrightarrow \\
H^{i+1}(A^\bullet) \longrightarrow \cdots
\end{array}$$

3. $\operatorname{Hom}(X, \cdot)$ *commutes with limits.* Suppose A_i ($i \in \mathcal{I}$) is a diagram in $\mathcal{D}$ indexed by $\mathcal{I}$, and $\varprojlim A_i \to A_i$ is its limit. Then for any $X \in \mathcal{D}$, $\operatorname{Hom}(X, \varprojlim A_i) \to \operatorname{Hom}(X, A_i)$ is the limit $\varprojlim \operatorname{Hom}(X, A_i)$.

4. *(for those familiar with differentiable functions)* In the "motivating example" of the sheaf of differentiable functions, show that $\mathfrak{m}_x$ is the only maximal ideal of $\mathcal{O}_x$.

5-. *"A presheaf is the same as a contravariant functor"* Given any topological space X, we can get a category, called the "category of open sets" (discussed last week), where the objects are the open sets and the morphisms are inclusions. Verify that the data of a presheaf is precisely the data of a contravariant functor from the category of open sets of X to the category of sets. (This interpretation is suprisingly useful.)

6-. *(unimportant exercise for category-lovers)* The gluability axiom may be interpreted as saying that $\mathcal{F}(\cup_{i \in I} U_i)$ is a certain limit. What is that limit?

7. *(important Exercise: constant presheaf and locally constant sheaf*
(a) Let X be a topological space, and S a set with more than one element, and define $\mathcal{F}(U) = S$ for all open sets U. Show that this forms a presheaf (with the obvious restriction maps), and even satisfies the identity axiom. We denote this presheaf $\underline{S}^{pre}$. Show that this needn't form a sheaf. This is called the *constant presheaf with values in* S.
(b) Now let $\mathcal{F}(U)$ be the maps to S that are *locally constant*, i.e. for any point x in U, there is a neighborhood of x where the function is constant. Show that this is a *sheaf*. (A better

description is this: endow S with the discrete topology, and let $\mathcal{F}(U)$ be the continuous maps $U \to S$. Using this description, this follows immediately from Exercise 9 below.) We will call this the *locally constant sheaf*. This is usually called the *constant sheaf*.

8-. *(more examples of presheaves that are not sheaves)* Show that the following are presheaves on $\mathbb{C}$ (with the usual topology), but not sheaves: (a) bounded functions, (b) holomorphic functions admitting a holomorphic square root.

9. Suppose Y is a topological space. Show that "continuous maps to Y" form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps to Y. Show that this forms a sheaf.

10. This is a fancier example of the previous exercise.
(a) Suppose we are given a continuous map $f : Y \to X$. Show that "sections of f" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps s to Y such that $f \circ s = id|_U$. Show that this forms a sheaf. (For those who have heard of vector bundles, these are a good example.)
(b) (This exercise is for those who know topological group is. If you don't know what a topological group is, you might be able to guess.) Suppose that Y is a topological group. Show that maps to Y form a sheaf of *groups*. (A special case turned up in class.)

11. *(important exercise: the direct image sheaf or pushforward sheaf)* Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X. Then define $f_*\mathcal{F}$ by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$, where V is an open subset of Y. Show that $f_*\mathcal{F}$ is a sheaf. This is called a *direct image sheaf* of *pushforward sheaf*. More precisely, $f_*\mathcal{F}$ is called the *pushforward of $\mathcal{F}$ by* f.

12. *(pushforward induces maps of stalks)* Suppose $\mathcal{F}$ is a sheaf of sets (or rings or A-modules). If $f(x) = y$, describe the natural morphism of stalks $(f_*\mathcal{F})_y \to \mathcal{F}_x$. (You can use the explicit definition of stalk using representatives, or the universal property. If you prefer one way, you should try the other.)

Class 4.

13. Suppose $f : X \to Y$ is a continuous map of topological spaces (i.e. a morphism in the category of topological spaces). Show that pushforward gives a functor from $\{$ sheaves of sets on $X \}$ to $\{$ sheaves of sets on $Y \}$. Here "sets" can be replaced by any category.

14. *(important exercise and definition: "Sheaf* Hom*")* Suppose $\mathcal{F}$ and $\mathcal{G}$ are two sheaves on X. (In fact, it will suffice that $\mathcal{F}$ is a presheaf.) Let $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ be the collection of data

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U).$$

(Recall the notation $\mathcal{F}|_U$, the restriction of the sheaf to the open set U, see last day's notes.) Show that this is a sheaf. This is called the "sheaf $\underline{\mathrm{Hom}}$". Show that if $\mathcal{G}$ is a sheaf of abelian groups, then $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a sheaf of abelian groups.

15. Show that $\ker_{\text{pre}} f$ is a presheaf. (Hint: if $U \hookrightarrow V$, there is a natural map $\text{res}_{V,U} : \mathcal{G}(V)/f_V(\mathcal{F}(V)) \to \mathcal{G}(U)/f_U(\mathcal{F}(U))$ by chasing the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker_{\text{pre}} f_V & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \\ & & \Big\downarrow{\scriptstyle \exists!} & & \Big\downarrow{\scriptstyle \text{res}_{V,U}} & & \Big\downarrow{\scriptstyle \text{res}_{V,U}} \\ 0 & \longrightarrow & \ker_{\text{pre}} f_U & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \end{array}$$

You should check that the restriction maps compose as desired.)

16. *(the cokernel deserves its name)* Show that the presheaf cokernel satisfies the universal property of cokernels in the category of presheaves.

17. If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$ is an exact sequence of presheaves of abelian groups, then $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \cdots \to \mathcal{F}_n(U) \to 0$ is also an exact sequence for all U, and vice versa.

18. *(important)* Suppose $f : \mathcal{F} \to \mathcal{G}$ is a morphism of *sheaves*. Show that the presheaf kernel $\ker_{\text{pre}} f$ is in fact a sheaf. Show that it satisfies the universal property of kernels. (Hint: the second question follows immediately from the fact that $\ker_{\text{pre}} f$ satisfies the universal property in the category of *presheaves*.)

19. *(important exercise)* Let X be $\mathbb{C}$ with the classical topology, let $\underline{\mathbb{Z}}$ be the locally constant sheaf on X with group $\mathbb{Z}$, $\mathcal{O}_X$ the sheaf of holomorphic functions, and $\mathcal{F}$ the *presheaf* of functions admitting a holomorphic logarithm. (Why is $\mathcal{F}$ not a sheaf?) Show that

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{f \mapsto \exp 2\pi i f} \mathcal{F} \longrightarrow 0$$

where $\underline{\mathbb{Z}} \to \mathcal{O}_X$ is the natural inclusion. Show that this is an exact sequence *of presheaves*. Show that $\mathcal{F}$ is *not* a sheaf. (Hint: $\mathcal{F}$ does not satisfy the gluability axiom. The problem is that there are functions that don't have a logarithm that locally have a logarithm.)

20+. *(important exercise: sections are determined by stalks)* Prove that a section of a sheaf is determined by its germs, i.e. the natural map

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x \tag{1}$$

is injective. (Hint # 1: you won't use the gluability axiom, so this is true for separated presheaves. Hint # 2: it is false for presheaves in general, see Exercise , so you *will* use the identity axiom.)

21+. *(important)* Prove that any choice of compatible germs for $\mathcal{F}$ over U is the image of a section of $\mathcal{F}$ over U. (Hint: you will use gluability.)

22. Show a morphism of (pre)sheaves (of sets, or rings, or abelian groups, or $\mathcal{O}_X$-modules) induces a morphism of stalks. More precisely, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of (pre)sheaves on X, and $x \in X$, describe a natural map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$.

23. *(morphisms are determined by stalks)* Show that morphisms of sheaves are determined by morphisms of stalks. Hint: consider the following diagram.

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \big\downarrow & & \big\downarrow \\ \prod_{x\in U}\mathcal{F}_x & \longrightarrow & \prod_{x\in U}\mathcal{G}_x \end{array} \tag{2}$$

24. *(tricky: isomorphisms are determined by stalks)* Show that a morphism of sheaves is an isomorphism if and only if it induces an isomorphism of all stalks. (Hint: Use (2). Injectivity uses the previous exercise . Surjectivity will use gluability, and is more subtle.)

25. Problems 20, 21, 23, and 24 are all false for general presheaves. Give counterexamples to three of them. (General hint for finding counterexamples of this sort: consider a 2-point space with the discrete topology, i.e. every subset is open.)

26-. Show that sheafification (as defined by universal property) is unique up to unique isomorphism. Show that if $\mathcal{F}$ is a sheaf, then the sheafification is $\mathcal{F} \xrightarrow{\text{id}} \mathcal{F}$.

27. Show that $\mathcal{F}^{sh}$ (using the tautological restriction maps) forms a sheaf.

28-. Describe a natural map $sh : \mathcal{F} \to \mathcal{F}^{sh}$.

29+. Show that the map sh satisfies the universal property of sheafification.

30. Use the universal property to show that for any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$. Show that sheafification is a functor from presheaves to sheaves.

31+. *(useful exercise for category-lovers)* Show that the sheafification functor is left-adjoint to the forgetful functor from sheaves on X to presheaves on X.

32. Show $\mathcal{F} \to \mathcal{F}^{sh}$ induces an isomorphism of stalks. (Possible hint: Use the concrete description of the stalks. Another possibility: judicious use of adjoints.)

33+. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (of sets) on at topological space X. Show that the following are equivalent.

(a) ϕ is a monomorphism in the category of sheaves.
(b) ϕ is injective on the level of stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ injective for all $x \in X$.
(c) ϕ is injective on the level of open sets: $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open $U \subset X$.

(Possible hints: for (b) implies (a), recall that morphisms are determined by stalks, Exercise . For (a) implies (b), judiciously choose a skyscraper sheaf. For (a) implies (c), judiciously the "indicator sheaf" with one section over every open set contained in U, and no section over any other open set.)

34. Continuing the notation of the previous exercise, show that the following are equivalent.

(a) ϕ is a epimorphism in the category of sheaves.
(b) ϕ is surjective on the level of stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ surjective for all $x \in X$.

35. Show that $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$ describes $\mathcal{O}_X^*$ as a quotient sheaf of $\mathcal{O}_X$. Show that it is not surjective on all open sets.

36. Show that the stalk of the kernel is the kernel of the stalks: there is a natural isomorphism

$$(\ker(\mathcal{F} \to \mathcal{G}))_x \cong \ker(\mathcal{F}_x \to \mathcal{G}_x).$$

37. Show that the stalk of the cokernel is naturally isomorphic to the cokernel of the stalk.

38. *(Left-exactness of the global section functor)* Suppose $U \subset X$ is an open set, and $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves of abelian groups. Show that

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact. Give an example to show that the global section functor is not exact. (Hint: the exponential exact sequence.)

39+. *(Left-exactness of pushforward)* Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves of abelian groups on X. If $f : X \to Y$ is a continuous map, show that

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H}$$

is exact. (The previous exercise, dealing with the left-exactness of the global section functor can be interpreted as a special case of this, in the case where Y is a point.)

40. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups. Show that the image sheaf $\operatorname{im}\phi$ is the sheafification of the image presheaf. (You must use the definition of image in an abelian category. In fact, this gives the accepted definition of image sheaf for a morphism of sheaves of sets.)

41. Show that if $(X, \mathcal{O}_X)$ is a ringed space, then $\mathcal{O}_X$-modules form an abelian category. (There isn't much more to check!)

42. *(important exercise: tensor products of $\mathcal{O}_X$-modules)* (a) Suppose $\mathcal{O}_X$ is a sheaf of rings on X. Define (categorically) what we should mean by tensor product of two $\mathcal{O}_X$-modules. Give an explicit construction, and show that it satisfies your categorical definition. *Hint:* take the "presheaf tensor product" — which needs to be defined — and sheafify. Note: $\otimes_{\mathcal{O}_X}$ is often written $\otimes$ when the subscript is clear from the context.
(b) Show that the tensor product of stalks is the stalk of tensor product.

E-mail address: `vakil@math.stanford.edu`