FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 50

RAVI VAKIL

CONTENTS

1. DIFFERENTIALS: MOTIVATION AND GAME PLAN

Differentials are an intuitive geometric notion, and we're going to figure out the right description of them algebraically. I find the algebraic manifestation a little non-intuitive, so I always like to tie it to the geometry. So please don't tune out of the statements. Also, I want you to notice that although the algebraic statements are odd, none of the proofs are hard or long. You'll notice that this topic could have been done as soon as we knew about morphisms and quasicoherent sheaves.

I prefer to introduce new ideas with a number of examples, but in this case I'm going to spend a fair amount of time discussing theory, and only then get to a number of examples.

Suppose X is a "smooth" k-variety. We intend to define a tangent bundle. We'll see that the right way to do this will easily apply in much more general circumstances.

• We'll see that cotangent is more "natural" for schemes than tangent bundle. This is similar to the fact that the Zariski *cotangent* space is more natural than the *tangent space* (i.e. if A is a ring and $\mathfrak{m}$ is a maximal ideal, then $\mathfrak{m}/\mathfrak{m}^2$ is "more natural" than $(\mathfrak{m}/\mathfrak{m}^2)^\vee$). Both of these notions are because we are understanding "spaces" via their (sheaf of) functions on them, which is somehow dual to the geometric pictures you have of spaces in your mind.

So we'll define the cotangent sheaf first. An element of the (co)tangent space will be called a **(co)tangent vector**.

• Our construction will automatically apply for general X, even if X is not "smooth" (or even at all nice, e.g. finite type). The cotangent sheaf won't be locally free, but it will still be a quasicoherent sheaf.

• Better yet, this construction will naturally work "relatively". For *any* $X \to Y$, we'll define $\Omega_{X/Y}$, a quasicoherent sheaf on X, the sheaf of *relative differentials*. The fiber of this sheaf

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at a point will be the cotangent vectors of the fiber of the map. This will specialize to the earlier case by taking $Y = \operatorname{Spec} k$. The idea is that this glues together the cotangent sheaves of the fibers of the family. Figure 1 is a sketch of the relative tangent space of a map $X \to Y$ at a point $p \in X$ — it is the tangent to the fiber. (The tangent space is easier to draw than the cotangent space!) An element of the relative (co)tangent space is called a **vertical** or **relative (co)tangent vector**.

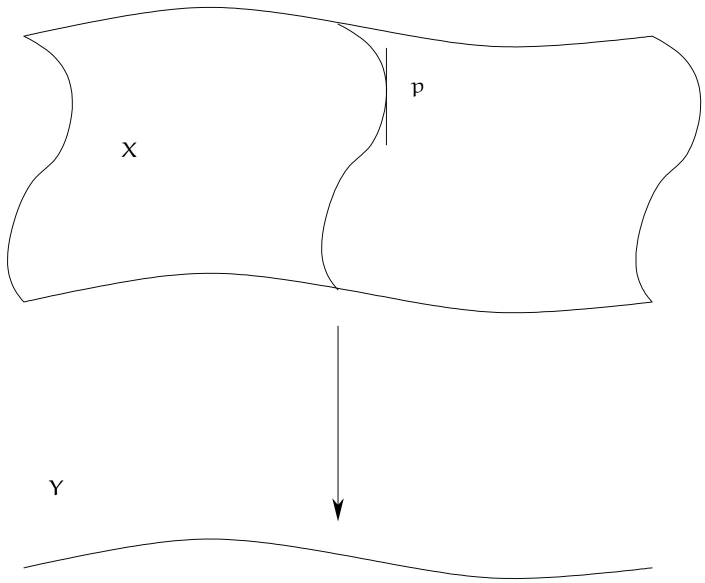

FIGURE 1. The relative tangent space of a morphism $X \to Y$ at a point p

2. THE AFFINE CASE: TWO OF THREE DEFINITIONS

We'll first study the affine case. Suppose A is a B-algebra, so we have a morphism of rings $\phi : B \to A$ and a morphism of schemes $\operatorname{Spec} A \to \operatorname{Spec} B$. I will define an A-module $\Omega_{A/B}$ in three ways. This is called the **module of relative differentials** or the **module of Kähler differentials**. The module of differentials will be defined to be this module, as well as a map $d : A \to \Omega_{A/B}$ satisfying three properties.

(i) **additivity.** $da + da' = d(a + a')$
(ii) **Leibniz.** $d(aa') = a\ da' + \ a'da$
(iii) **triviality on pullbacks.** $db = 0$ for $b \in \phi(B)$.

2.A. TRIVIAL EXERCISE. Show that d is B-linear. (In general it will not be A-linear.)

2.B. EXERCISE. Prove the quotient rule: if $b = as$, then $da = (s\ db - b\ ds)/s^2$.

2.C. EXERCISE. State and prove the chain rule for $d(f(g))$ where f is a polynomial with B-coefficients, and $g \in A$. (As motivation, think of the case $B = k$. So for example, $da^n = na^{n-1}da$, and more generally, if f is a polynomial in one variable, $df(a) = f'(a)\ da$, where f' is defined formally: if $f = \sum c_i x^i$ then $f' = \sum c_i i x^{i-1}$.)

I'll give you three definitions of the module of Kähler differentials, which will soon "sheafifiy" to the sheaf of relative differentials. The first definition is a concrete hands-on definition. The second is by universal property. And the third will globalize well, and will allow us to define $\Omega_{X/Y}$ conveniently in general.

2.1. First definition of differentials: explicit description. We define $\Omega_{A/B}$ to be finite A-linear combinations of symbols "da" for $a \in A$, subject to the three rules (i)–(iii) above. For example, take $A = k[x, y]$, $B = k$. Then a sample differential is $3x^2\ dy + 4\ dx \in \Omega_{A/B}$. We have identities such as $d(3xy^2) = 3y^2\ dx + 6xy\ dy$.

Key fact. Note that if A is generated over B (as an algebra) by $x_i \in A$ (where i lies in some index set, possibly infinite), subject to some relations r_j (where j lies in some index set, and each is a polynomial in the x_i), then the A-module $\Omega_{A/B}$ is generated by the dx_i, subject to the relations (i)—(iii) and $dr_j = 0$. In short, we needn't take every single element of A; we can take a generating set. And we needn't take every single relation among these generating elements; we can take generators of the relations.

2.D. EXERCISE. Verify the above key fact.

In particular:

2.2. *Proposition. — If A is a finitely generated B-algebra, then $\Omega_{A/B}$ is a finite type (=finitely generated) A-module. If A is a finitely presented B-algebra, then $\Omega_{A/B}$ is a finitely presented A-module.*

An algebra A is **finitely presented** over another algebra B if it can be expressed with finite number of generators (=finite type) and finite number of relations:

$$A = B[x_1, \ldots, x_n]/(r_1(x_1, \ldots, x_n), \ldots, r_j(x_1, \ldots, x_n)).$$

If A is Noetherian, then the two hypotheses are the same, so most of you will not care.)

Let's now see some examples. Among these examples are three particularly important kinds of ring maps that we often consider: adding free variables; localizing; and taking quotients. If we know how to deal with these, we know (at least in theory) how to deal with any ring map.

2.3. Example: taking a quotient. If $A = B/I$, then $\Omega_{A/B} = 0$ basically immediately: $da = 0$ for all $a \in A$, as each such a is the image of an element of B. This should be believable; in this case, there are no "vertical tangent vectors".

2.4. Example: adding variables. If $A = B[x_1, \dots, x_n]$, then $\Omega_{A/B} = A dx_1 \oplus \cdots \oplus A dx_n$. (Note that this argument applies even if we add an arbitrarily infinite number of indeterminates.) The intuitive geometry behind this makes the answer very reasonable. The cotangent bundle should indeed be trivial of rank n.

2.5. Example: two variables and one relation. If $B = \mathbb{C}$, and $A = \mathbb{C}[x,y]/(y^2 - x^3)$, then $\Omega_{A/B} = (A\ dx \oplus A\ dy)/(2y\ dy - 3x^2\ dx)$.

2.6. Example: localization. If S is a multiplicative set of B, and $A = S^{-1}B$, then $\Omega_{A/B} = 0$. Reason: the quotient rule holds, Exercise 2.B, so if $a = b/s$, then $da = (s\ db - b\ ds)/s^2 = 0$. If $A = B_f$ for example, this is intuitively believable; then $\operatorname{Spec} A$ is an open subset of $\operatorname{Spec} B$, so there should be no vertical (co)tangent vectors.

2.E. EXERCISE. Suppose k is a field, and K is a separable algebraic extension of k. Show that $\Omega_{K/k} = 0$. (Warning: do not assume that K/k is a finite extension!)

2.7. *Exercise (Jacobian description of $\Omega_{A/B}$). —* Suppose $A = B[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Then $\Omega_{A/B} = \{\oplus_i A dx_i\}/\{df_j = 0\}$ maybe interpreted as the cokernel of the Jacobian matrix $J : A^{\oplus r} \to A^{\oplus n}$.

I now want to tell you two handy (geometrically motivated) exact sequences. The arguments are a bit tricky. They are useful, but a little less useful than the foundation facts above.

2.8. *Theorem (relative cotangent sequence, affine version). — Suppose $C \to B \to A$ are ring homomorphisms. Then there is a natural exact sequence of A-modules*

$$A \otimes_B \Omega_{B/C} \to \Omega_{A/C} \to \Omega_{A/B} \to 0.$$

The proof will be quite straightforward algebraically, but the statement comes fundamentally from geometry, and that is how I remember it. Figure 2 is a sketch of a map $X \xrightarrow{f} Y$. Here X should be interpreted as $\operatorname{Spec} A$, Y as $\operatorname{Spec} B$, and $\operatorname{Spec} C$ is a point. (If you would like a picture with a higher-dimensional $\operatorname{Spec} C$, just take the "product" of Figure 2 with a curve.) In the Figure, Y is "smooth", and X is "smooth over Y" — roughly, all fibers are smooth. p is a point of X. Then the tangent space of the fiber of f at p is certainly a subspace of the tangent space of the total space of X at p. The cokernel is naturally the pullback of the tangent space of Y at $f(p)$. This short exact sequence for each p should be part of a short exact sequence of sheaves

$$0 \to T_{X/Y} \to T_{X/Z} \to f^*T_{Y/Z} \to 0$$

on X Dualizing this yields

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

This is precisely the statement of the Theorem, except we also have left-exactness. This discrepancy is because the statement of the theorem is more general; we'll may later see that in the "smooth" case, we'll indeed have left-exactness.

2.9. *Unimportant aside.* As always, whenever you see something right-exact, you should suspect that there should be some sort of (co)homology theory so that this is the end of a long exact sequence. This is indeed the case, and this exact sequence involves *André-Quillen homology*. You should expect that the next term to the left should be the first homology corresponding to A/B, and in particular shouldn't involve C. So if you already suspect that you have exactness on the left in the case where A/B and B/C are "smooth" (whatever that means), and the intuition of Figure 2 applies, then you should expect further that all that is necessary is that A/B be "smooth", and that this would imply that the first André-Quillen homology should be zero. Even though you wouldn't precisely know what all the words meant, you would be completely correct!

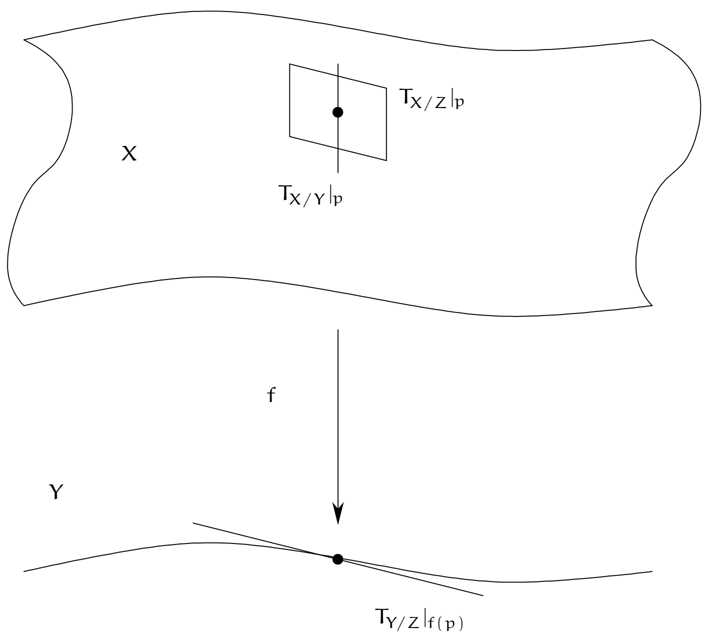

FIGURE 2. A sketch of the geometry behind the relative cotangent sequence

Proof of the relative cotangent sequence (affine version) 2.8.

First, note that surjectivity of $\Omega_{A/C} \to \Omega_{A/B}$ is clear, as this map is given by $da \mapsto da$ $(a \in A)$.

Next, the composition over the middle term is clearly 0, as this composition is given by $db \to db \to 0$.

Finally, we wish to identify $\Omega_{A/B}$ as the cokernel of $A \otimes_B \Omega_{B/C} \to \Omega_{A/C}$. Now $\Omega_{A/B}$ is exactly the same as $\Omega_{A/C}$, except we have extra relations: $db = 0$ for $b \in B$. These are precisely the images of $1 \otimes db$ on the left. $\square$

2.10. *Theorem (conormal exact sequence, affine version). — Suppose* B *is a* C*-algebra,* I *is an ideal of* B*, and* $A = B/I$*. Then there is a natural exact sequence of* A*-modules*

$$I/I^2 \xrightarrow{\delta : i \mapsto 1 \otimes di} A \otimes_B \Omega_{B/C} \xrightarrow{a \otimes db \mapsto a\, db} \Omega_{A/C} \longrightarrow 0.$$

Before getting to the proof, some discussion may be helpful. First, the map δ needs to be rigorously defined. It is the map $1 \otimes d : B/I \otimes_B I \to B/I \otimes \Omega_{B/C}$.

As with the relative cotangent sequence, the conormal exact sequence is fundamentally about geometry. To motivate it, consider the sketch of Figure 3. In the sketch, everything is "smooth", X is one-dimensional, Y is two-dimensional, j is the inclusion $j : X \hookrightarrow Y$, and Z (omitted) is a point. Then at a point $p \in X$, the tangent space $T_X|_p$ clearly injects into the tangent space of $j(p)$ in Y, and the cokernel is the normal vector space to X in Y at p. This should give an exact sequence of bundles on X:

$$0 \to T_X \to j^* T_Y \to N_{X/Y} \to 0.$$

dualizing this should give

$$0 \to N^{\vee}_{X/Y} \to j^* \Omega_Y / Z \to \Omega_X / Z \to 0.$$

This is precisely what appears in the statement of the Theorem, except we see I/I^2 rather than $N^{\vee}_{\operatorname{Spec} A / \operatorname{Spec} B}$, and the exact sequence in algebraic geometry is not necessary exact on the left.

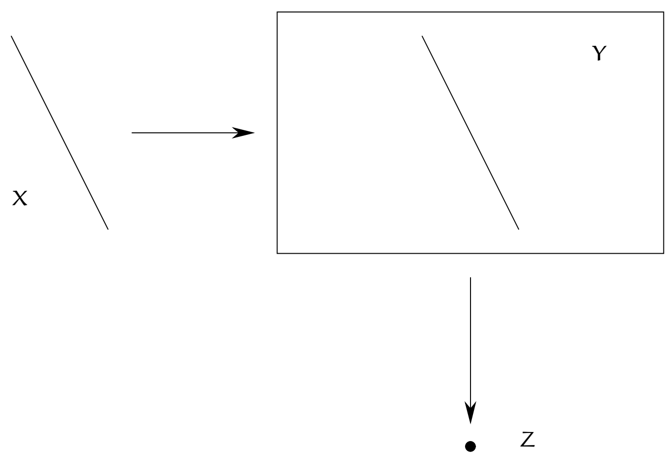

FIGURE 3. A sketch of the geometry behind the conormal exact sequence

2.11. We resolve the first issue by declaring I/I^2 to be the *conormal module*, and indeed we'll soon see the obvious analogue as the *conormal sheaf*. (Further evidence that I/I^2 deserves to be called the conormal bundle: if $\operatorname{Spec} A$ is a closed point of $\operatorname{Spec} B$, we expect the conormal space to be precisely the cotangent space. And indeed if $A = B/\mathfrak{m}$, the Zariski cotangent space is $\mathfrak{m}/\mathfrak{m}^2$.)

And we resolve the second by expecting that the seqeunce of Theorem 2.10 is exact on the left if X/Y and Y/Z (and hence X/Z) are "smooth" whatever that means. This is indeed the case. (If you enjoyed Remark 2.9, you might correctly guess several things. The next term on the left should be the André-Quillen homology of A/C, so we should only need that A/C is smooth, and B should be irrelevant. Also, if $A = B/I$, then we should expect that I/I^2 is the first André-Quillen homology of A/B.)

Proof of the conormal exact sequence (affine version) 2.10. We need to identify the cokernel of $\delta : I/I^2 \to A \otimes_B \Omega_{B/C}$ with $\Omega_{A/C}$. Consider $A \otimes_B \Omega_{B/C}$. As an A-module, it is generated by db ($b \in B$), subject to three relations: $dc = 0$ for $c \in \phi(C)$ (where $\phi : C \to B$ describes B as a C-algebra), additivity, and the Leibniz rule. Given any relation *in* B, d of that relation is 0.

Now $\Omega_{A/C}$ is defined similarly, except there are more relations *in* A; these are precisely the elements of $i \in B$. Thus we obtain $\Omega_{A/C}$ by starting out with $A \otimes_B \Omega_{B/C}$, and adding the additional relations di where $i \in I$. But this is precisely the image of δ! $\square$

2.12. Second definition: universal property. Here is a second definition that is important philosophically, by universal property. Technically, it isn't a definition: by universal property nonsense, it shows that if the module exists (with the d map), then it is unique up to unique isomorphism, and then one still has to construct it to make sure that it exists.

Suppose A is a B-algebra, and M is a A-module. An B-*linear derivation of* A *into* M is a map $d : A \to M$ of B-modules (*not necessarily* A-*modules*) satisfying the Leibniz rule: $d(fg) = f\ dg + g\ df$. As an example, suppose $B = k$, and $A = k[x]$, and $M = A$. Then an example of a k-linear derivation is d/dx. As a second example, if $B = k$, $A = k[x]$, and $M = k$. Then an example of a k-linear derivation is $d/dx|_0$.

Then $d : A \to \Omega_{A/B}$ is defined by the following universal property: any other B-linear derivation $d' : A \to M$ factors uniquely through d:

$$\begin{tikzcd} A \arrow[rr, "d'"] \arrow[dr, "d"'] & & M \\ & \Omega_{A/B} \arrow[ur, "f"'] & \end{tikzcd}$$

Here f is a map of A-modules. (Note again that d and d' are not! They are only B-linear.) By universal property nonsense, if it exists, it is unique up to unique isomorphism. The candidate I described earlier clearly satisfies this universal property (in particular, it is a derivation!), hence this is it. [Thus Ω is the "unversal derivation". I should rewrite this paragraph at some point. Justin points out: the map defined earlier is a derivation, but I never really say that; thus the original map, together with Ω, is a universal derivation.]

The next result will give you more evidence that this deserves to be called the (relative) cotangent bundle.

2.13. *Proposition.* *Suppose* B *is a* k*-algebra, with residue field* k*. Then the natural map* $\delta : \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B k$ *is an isomorphism.*

Proof. By the conormal exact sequence 2.10 with $I = \mathfrak{m}$ and $A = C = k$, δ is a surjection (as $\Omega_{k/k} = 0$), so we need to show that it is injection, or equivalently that $\mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is a surjection. But any element on the right is indeed a derivation from B to k (an earlier exercise from back in the dark ages on the Zariski tangent space), which is precisely an element of $\mathrm{Hom}_B(\Omega_{B/k}, k)$ (by the universal property of $\Omega_{B/k}$), which is canonically isomorphic to $\mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k)$ as desired. $\square$

Remark. As a corollary, this (in combination with the Jacobian exercise 2.7 above) gives a second proof of an exercise from the first quarter, showing the Jacobian criterion for nonsingular varieties over an algebraically closed field.

Depending on how your brain works, you may prefer using the first (constructive) or second (universal property) definition to do the next two exercises.

2.F. EXERCISE. (a) *(pullback of differentials)* If

$$\begin{tikzcd} A' & A \arrow[l] \\ B' \arrow[u] & B \arrow[l] \arrow[u] \end{tikzcd}$$

is a commutative diagram, show that there is a natural homomorphism of A'-modules $A' \otimes_A \Omega_{A/B} \to \Omega_{A'/B'}$. An important special case is $B = B'$.
(b) *(differentials behave well with respect to base extension, affine case)* If furthermore the above diagram is a tensor diagram (i.e. $A' \cong B' \otimes_B A$) then show that $A' \otimes_A \Omega_{A/B} \to \Omega_{A'/B'}$ is an isomorphism.

2.G. EXERCISE: LOCALIZATION (STRONGER FORM). If S is a multiplicative set of A, show that there is a natural isomorphism $\Omega_{S^{-1}A/B} \cong S^{-1}\Omega_{A/B}$. (Again, this should be believable from the intuitive picture of "vertical cotangent vectors".) If T is a multiplicative set of B, show that there is a natural isomorphism $\Omega_{S^{-1}A/T^{-1}B} \cong S^{-1}\Omega_{A/B}$ where S is the multiplicative set of A that is the image of the multiplicative set $T \subset B$. [Ziyu used the relative cotangent sequence.]

E-mail address: vakil@math.stanford.edu