FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 48

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CONTENTS

1. A LITTLE MORE ABOUT CUBIC PLANE CURVES

1.A. IMPORTANT EXERCISE: A DEGENERATE ELLIPTIC CURVE. Consider the genus 1 curve $C \subset \mathbb{P}^2_k$ given by $y^2z = x^3 + x^2z$, with the point $p = [0;1;0]$. Emulate the above argument to show that $C - [0;0;1]$ is a group variety. Show that it is isomorphic to $\mathbb{G}_m$ (the multiplicative group) with co-ordinate $t = y/x$, by showing an isomorphism of schemes, and showing that multiplication and inverse in both group varieties agree under this isomorphism.

1.B. EXERCISE: AN EVEN MORE DEGENERATE ELLIPTIC CURVE. Consider the genus 1 curve $C \subset \mathbb{P}^2_k$ given by $y^2z = x^3$, with the point $p = [0;1;0]$. Emulate the above argument to show that $C - [0;0;1]$ is a group variety. Show that it is isomorphic to $\mathbb{A}^1$ (with additive group structure) with co-ordinate $t = y/x$, by showing an isomorphism of schemes, and showing that multiplication/addition and inverse in both group varieties agree under this isomorphism.

I then gave proofs of Pappas' Theorem and Pascal's "Mystical Hexagon" theorem.

2. LINE BUNDLES OF DEGREE 4, AND PONCELET'S PORISM

The story doesn't stop in degree 3. In the same way that we showed that a canonically embedded nonhyperelliptic curve of genus 4 is the complete intersection in $\mathbb{P}^3_k$ of a quadric and a cubic, we can show the following.

2.A. EXERCISE. Show that the complete linear system for $\mathcal{O}(4p)$ embeds E in $\mathbb{P}^3$ as the complete intersection of two quadrics. (Hint: Show the image of E is contained in at least

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2 linearly independent quadrics. Show that neither can be reducible, so they share no components. Use Bezout's theorem.)

We can use this to prove a beautiful fact in classical geometry: Poncelet's porism. Suppose C and D are two ellipses in the real plane, with C containing D. Choose any point p_0 on C. Choose one of the two tangents ℓ_1 from p to D. Then ℓ_1 meets C at two points in total: p_0 and another points p_1. From p_1, there are two tangents to D, ℓ_1 and another line ℓ_2. The line ℓ_2 meets C at some other point p_2. Continue this to get a sequence of points $p_0, p_1, p_2, \ldots$. Suppose this sequence starting with p_0 is periodic, i.e. $p_0 = p_n$ for some n. Then it is periodic with *any* starting point $p \in C$. I drew a picture of this.

Let's see what this has to do with elliptic curves. We work over the complex numbers and at the end consider what our results over the real numbers. For the rest of this discussion, we assume that k is an algebraically closed field of characteristic not 2.

2.B. EXERCISE. Suppose D is a nonsingular conic in the plane $\mathbb{P}^2_k$. Suppose p is a point on the plane not on D. Then there are precisely 2 tangents to D containing p.

Thus we have verified one of the implicit statements in the set-up for Poncelet's porism.

Next, suppose Q is a nonsingular quadric in $\mathbb{P}^3$, and q is a point not on Q. Then the projection from q to $\mathbb{P}^2$ describes Q as a branched double-cover of $\mathbb{P}^2$. We should be explicit about what we mean about "branching": the lines through q correspond to the (closed) points of $\mathbb{P}^2$. Most lines meet Q in 2 points. The branch points in $\mathbb{P}^2$ correspond to those that meet Q in only one point (with multiplicity 2 of course).

2.C. EXERCISE. Show that this double cover is branched over a nonsingular conic D in $\mathbb{P}^2$. (If it helps, choose explicit co-ordinates.)

Side remark: we have stated earlier that $\operatorname{Pic}(\mathbb{P}^2 - D) \cong \mathbb{Z}/2$, and that this was related to the fact that (over the complex numbers) $\pi_1(\mathbb{P}^2 - C) = \mathbb{Z}/2$. This latter fact implies that the universal cover of $\mathbb{P}^2 - C$ is a double cover. We have now produced the double cover: the quadric Q minus the branch divisor. We can even use this to prove that $\pi_1(\mathbb{P}^2 - C) = \mathbb{Z}/2$: please ask me for the short argument.

Since Q is a nonsingular quadric over an algebraically closed field of characteristic not 2, $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, and has two rulings. What are the images of the lines in each ruling in $\mathbb{P}^2$? Suppose ℓ is a line in $\mathbb{P}^2$. Then the preimage of ℓ in $\mathbb{P}^3$ is a plane Π in $\mathbb{P}^3$ containing q. Q meets this plane Π in a conic.

If this conic $Q \cap \Pi$ is nonsingular, then we are precisely in the situation of Exercise 2.B, and this may help you see that the degree of the branch curve is 2 (Exercise 2.C). Also, since $Q \cap \Pi$ is not singular at any point (i.e. the germ of the equation of $Q \cap P$ at any point r is not contained in the square of the maximal ideal at r), $Q \cap \Pi$ is not a tangent plane to Q.

On the other hand, if $Q \cap \Pi$ is singular, then Π is a tangent plane to Q. And this singular conic is the union of two lines. (Why can't it be a double line?) Thus the two lines consist of one of each ruling.

Conversely, if l is a line in a ruling on Q, then the plane Π spanned by l and q must be tangent to Q: the conic $\Pi \cap Q$ contains a line and is thus singular.

We thus conclude that the image of any line on Q is a tangent line to D, and conversely the preimage of each tangent line on D is two lines on Q, one from each ruling.

We have recovered part of the picture of Poncelet: we have a nonsingular conic D in the plane. Let $C \subset \mathbb{P}^2$ be another conic in the plane, not tangent to D. Let $G \subset \mathbb{P}^3$ be the quadric surface that is the cone over C with vertex q. (Can you make this precise?)

2.D. EXERCISE. Show that $G \cap C$ is a nonsingular curve.

The complete intersection E of two quadric surfaces in $\mathbb{P}^3$ has genus 1. Choose any point of it, so E have an elliptic curve. By considering E as a subset of $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, we have two maps $E \to \mathbb{P}^1$, one corresponding to each factor. Both are degree 2. Let D_1 and D_2 be the two degree 2 divisors corresponding to these two double covers. If p_0' is a point of E, each ruling through p_0' meets E at one other point: the point $D_1 - p$ for the first ruling, and the point $D_2 - p$ for the second ruling. The image of p_0', the two lines, and $D_1 - p$ and $D_2 - p$ in the plane is a point p_0 in the plane, the two tangents to D from p_0, and the two points on C also on those two tangents.

Thus if we start at p_0', choose the other point of E on the line in the first ruling to obtain $p_1' = D_1 - p_0$, and then choose the other point of E through p_1' on a line in the second ruling, we obtain the point $p_2' = D_2 - p_1' = p_0' + (D_2 - D_1)$: a translation of p_0' by an amount independent of p_0'. Thus $p_{2n}' = p_0' + n(D_2 - D_1)$. In particular, if $p_{2n}' = p_0'$ for one choice of p_0', then this would still hold for *any* choice of p_0'.

2.E. EXERCISE. Put the above pieces together to prove Poncelet's porism.

3. FUN COUNTEREXAMPLES USING ELLIPTIC CURVES

We now give some fun counterexamples using our understanding of elliptic curves.

3.1. An example of a scheme that is locally factorial near a point p, but such that no affine open neighborhood of p has ring that is a Unique Factorization Domain.

Suppose E is an elliptic curve over $\mathbb{C}$ (or some other uncountable field). Consider $p \in E$. The local ring $\mathcal{O}_{E,p}$ is a Discrete Valuation Ring and hence a Unique Factorization Domain. Then an open neighborhood of E is of the form $E - q_1 - \cdots - q_n$. I claim that its Picard group is nontrivial. Recall the exact sequence:

$$\mathbb{Z}^n \xrightarrow{(a_1,\ldots,a_n)\mapsto a_1 q_1 + \cdots + a_n q_n} \operatorname{Pic} E \longrightarrow \operatorname{Pic}(E - q_1 - \cdots - q_n) \longrightarrow 0 .$$

But the group on the left is countable, and the group in the middle is uncountable, so the group on the right is non-zero.

3.2. Counterexamples using the existence of a non-torsion point.

We next give a number of counterexamples using the existence of a non-torsion point of a complex elliptic curve. We show the existence of such a point.

We have a "multiplication by n" map $[n] : E \to E$, which sends p to np. If $n = 0$, this has degree 0. If $n = 1$, it has degree 1. Given the complex picture of a torus, you might not be surprised that the degree of $\times n$ is n^2. If $n = 2$, we have almost shown that it has degree 4, as we have checked that there are precisely 4 points q such that $2p = 2q$. All that really shows is that the degree is at least 4. (We could check by hand that the degree is 4 is we really wanted to.)

3.3. *Proposition. — For each $n > 0$, the "multiplication by n" map has positive degree. In other words, there are only a finite number of n torsion points, and the $[n] \neq [0]$.*

Proof. We prove the result by induction; it is true for $n = 1$ and $n = 2$.

If n is odd, then assume otherwise that $nq = 0$ for all closed points q. Let r be a non-trivial 2-torsion point, so $2r = 0$. But $nr = 0$ as well, so $r = (n - 2[n/2])r = 0$, contradicting $r \neq 0$.

If n is even, then $[\times n] = [\times 2] \circ [\times(n/2)]$, and by our inductive hypothesis both $[\times 2]$ and $[\times(n/2)]$ have positive degree. □

In particular, the total number of torsion points on E is countable, so if k is an uncountable field, then E has an uncountable number of closed points (consider an open subset of the curve as $y^2 = x^3 + ax + b$; there are uncountably many choices for x, and each of them has 1 or 2 choices for y).

Thus *almost all* points on E are non-torsion. I'll use this to show you some pathologies.

3.4. An example of an affine open set that is not distinguished.

We can use this to see another example of an affine scheme X and an affine open subset Y that is not distinguished in X. (Our earlier example was $X = \mathbb{P}^2$ minus a conic, and $Y = X$ minus a line.) Let $X = E - p$, which is affine (easy, and an earlier exercise).

Let q be another point on E so that $q - p$ is non-torsion. Then $E - p - q$ is affine (we've shown all nonprojective nonsingular curves are affine). Assume that it is distinguished. Then there is a function f on $E - p$ that vanishes on q (to some positive order d). Thus

f is a rational function on E that vanishes at q to order d, and (as the total number of zeros minus poles of f is 0) has a pole at p of order d. But then $d(p-q) = 0$ in $\operatorname{Pic}^0 E$, contradicting our assumption that $p - q$ is non-torsion.

3.5. Example of variety with non-finitely-generated space of global sections.

We next show an example of a complex variety whose ring of global sections is not finitely generated. This is related to Hilbert's fourteenth problem, although I won't say how.

We begin with a preliminary exercise.

3.A. EXERCISE. Suppose X is a scheme, and L is the total space of a line bundle corresponding to invertible sheaf $\mathcal{L}$, so $L = \operatorname{Spec} \oplus_{n \geq 0} (\mathcal{L}^\vee)^{\otimes n}$. Show that $H^0(L, \mathcal{O}_L) = \oplus H^0(X, (\mathcal{L}^\vee)^{\otimes n})$.

Let E be an elliptic curve over some ground field k, N a degree 0 non-torsion invertible sheaf on E, and P a positive-degree invertible sheaf on E. Then $H^0(E, N^m \otimes P^n)$ is nonzero if and only if either (i) $n > 0$, or (ii) $m = n = 0$ (in which case the sections are elements of k). Thus the ring $R = \oplus_{m,n \geq 0} H^0(E, N^m \otimes P^n)$ is not finitely generated.

Now let X be the total space of the vector bundle $N \oplus P$ over E. Then the ring of global sections of X is R.

3.6. A proper nonprojective surface.

We finally sketch an example of a proper surface S over $\mathbb{C}$ that is not projective. We will see that the construction will work over uncountable fields, and (modulo an fact unproved here) $\mathbb{Q}$. We will construct it as a "fibration" $f : S \to C$ where C is a projective curve, and f is "locally projective", by which I mean that there is an open cover of C such that over each open set, f is projective. In particular, we will show that projectivity in the sense it is usually defined (without the data of a line bundle on the source, as we define it) is not a Zariski-local property.

As a result, we'll see some other interesting behavior, about the difficulty of gluing a scheme to itself (not typed up in the notes).

This is the simplest example I know. There are no examples of curves, as all proper curves are projective. This example is singular; in fact all proper nonsingular surfaces are projective.

Let C be two $\mathbb{P}^1$'s (C_1 and C_2) glued together at two points p and q, as shown in Figure 1. For example, consider a general conic union a line in $\mathbb{P}^2$. Clearly C is projective (over $\mathbb{C}$).

Let E be any complex elliptic curve, and r a non-torsion point on it. We construct an "E-bundle" over C as follows. over $C - p$, the family is trivial: $E \times (C - p)$. Similarly, over

$C - q$, the family is trivial. We glue these families together via the identity over C_1, and via translation by r over C_2. Call the resulting fibration $f : S \to C$.

Now E is proper, so f is proper over $C - p$ and $C - q$, and hence (by Zariski-locality of properness) f is a proper morphism. As C is proper, and properness is preserved by composition, S is a proper surface.

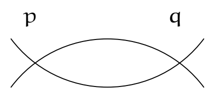

FIGURE 1. The $\mathbb{P}^1$'s glued together at two points

Suppose that S were projective, and that there was a closed immersion $S \to \mathbb{P}^n$ into projective space. Choose a hyperplane not containing the fiber of f over either p or q. This gives an effective Cartier divisor on S. Perhaps this effective Cartier divisor contains some fibers; if so, subtract them, to get another effective Cartier divisor containing no fibers. (There is no issue with subtracting these fibers, as away from the fibers over p and q, S is smooth, so on this locus, effective Weil divisors and effective Cartier divisors are the same.)

We will show that this is impossible.

I'll finish typing this in when I get a chance...

All we needed to make this argument work was the existence of a non-torsion point. Thus this argument works over any uncountable field. It also works over $\mathbb{Q}$ once one verifies that there is an elliptic curve over $\mathbb{Q}$ with a non-torsion point. This is a good excuse to mention the *Mordell-Weil Theorem*: for any elliptic curve E over $\mathbb{Q}$, the $\mathbb{Q}$-points of E form a *finitely generated* abelian group. By the classification of finitely generated abelian groups, the $\mathbb{Q}$-points are a direct sum of a torsion part, and of a free $\mathbb{Z}$-module. The rank of the $\mathbb{Z}$-module is called the *Mordell-Weil rank*. Thus this construction works once we have verified that there is an elliptic curve with positive Mordell-Weil rank.

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