This problem set is due on Friday, March 2 at Jarod Alper’s office door.

1. Prove the following fact used in our proof of the primitive element theorem. Suppose $F$ is an infinite field, and $V$ is a finite-dimensional vector space over $F$. Suppose $V_1, \ldots, V_n$ are proper subspaces of $V$ (i.e. $V_i \neq V$). Show that $\bigcup V_i \neq V$. Show that the statement is false without the hypothesis that $F$ is infinite.

2. Determine the Galois closure of the extension $\mathbb{Q}(\sqrt{1 + \sqrt{2}})/\mathbb{Q}$. What is its degree? (Dummit and Foote §14.4 problem 1)

3. Kummer generators for cyclic extensions. (Feel free to assume $n$ is prime.) Let $F$ be a field of characteristic not dividing $n$ containing the $n$th root of unity and let $K$ be a cyclic extension of degree $d$ dividing $n$. Then $K = F(\sqrt[n]{a})$ for some nonzero $a \in F$. Let $\sigma$ be a generator for the cyclic group $\text{Gal}(K/F)$.

   (a) Show that $\sigma(\sqrt[n]{a}) = \zeta \sqrt[n]{a}$ for some primitive $d$th root of unity $\zeta$.

   (b) Suppose $K = F(\sqrt[n]{a}) = F(\sqrt[n]{b})$. Use (a) to show that
   \[ \frac{\sigma(\sqrt[n]{a})}{\sqrt[n]{a}} = \left( \frac{\sigma(\sqrt[n]{b})}{\sqrt[n]{b}} \right)^i \]
   for some integer $i$ relatively prime to $d$. Conclude that $\sigma$ fixes the element $\sqrt[n]{\frac{a}{b}}$, so this is an element of $F$.

   (c) Prove that $K = F(\sqrt[n]{a}) = F(\sqrt[n]{b})$ if and only if $a = b^c c^n$ and $b = a^d d^n$ for some $c, d \in \mathbb{Z}$, i.e., if and only if $a$ and $b$ generate the same subgroup of $F^\times$ modulo $n$th powers. (Dummit and Foote §14.7, problem 7 — note as a special case that this classifies degree 2 extensions)

4. Prove that if $R$ is Noetherian, then so is the ring $R[[x]]$ of formal power series in the variable $x$ with coefficients from $R$. Hint: mimic the proof of the Hilbert basis theorem. (Dummit and Foote §15.1, problem 4)

Date: Friday, February 23, 2007.