This set is due at 6 p.m. on Monday January 29 in Jason Lo’s mailbox.

1. Show that \( \mathbb{Z}[i]/(7) \) is a field. Show that \( \mathbb{Z}[i]/(5) \) is not a field. Show that \( \mathbb{Z}[i]/(2) \) is not a field.

2. Show that \( p(x) = x^3 + 9x + 6 \) is irreducible in \( \mathbb{Q}[x] \). Let \( \alpha \) be a root of \( p(x) \). Find the inverse of \( 1 + \alpha \) in \( \mathbb{Q}(\alpha) \) (in the form \( a + b\alpha + c\alpha^2 \)).

3. Prove that if \( [F(\alpha) : F] \) is odd then \( F(\alpha) = F(\alpha^2) \).

4. Suppose \( F = \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \) where \( \alpha_i^3 \in \mathbb{Q} \) for \( i = 1, 2, \ldots, n \). Prove that \( \sqrt[3]{2} \notin F \).

5. Suppose \( E/F \) is a field extension, and \( E_1 \) and \( E_2 \) are two subextensions. Show that if \( E_1 \) and \( E_2 \) are finite (over \( F \)) then their compositum is finite. Show that if \( E_1 \) and \( E_2 \) are algebraic then their compositum is algebraic.

6. Find all intermediate fields in \( \mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q} \). Find the automorphism group of this field extension. (An automorphism of a field extension \( E/F \) is a field automorphism of \( E \) that preserves all the elements of the subfield \( F \).) Find all \( \alpha \) such that \( \mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \).