FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 20

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This set is due Thursday, May 25. You can hand it in to Rob Easton, in class or via his mailbox. It covers (roughly) classes 47 and 48.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1. *(for those who know what a Cohen-Macaulay scheme is)* Suppose $\pi : X \to Y$ is a map of locally Noetherian schemes, where both X and Y are equidimensional, and Y is nonsingular. Show that if any two of the following hold, then the third does as well:

- π is flat.
- X is Cohen-Macaulay.
- Every fiber X_y is Cohen-Macaulay of the expected dimension.

2. *(generated ⊗ generated = generated for finite type sheaves)* Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves on a scheme X that are generated by global sections. Show that $\mathcal{F} \otimes \mathcal{G}$ is also generated by global sections. In particular, if $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a scheme X, and both $\mathcal{L}$ and $\mathcal{M}$ are base-point-free, then so is $\mathcal{L} \otimes \mathcal{M}$. (This is often summarized as "base-point-free + base-point-free = base-point-free". The symbols $+$ is used rather than $\otimes$, because Pic is an abelian group.)

3. *(very ample + very ample = very ample)* If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a scheme X, and both $\mathcal{L}$ and $\mathcal{M}$ are base-point-free, then so is $\mathcal{L} \otimes \mathcal{M}$. Hint: Segre. In particular, tensor powers of a very ample invertible sheaf are very ample.

4+. *(very ample + relatively generated = very ample).* Suppose $\mathcal{L}$ is very ample, and $\mathcal{M}$ is relatively generated, both on $X \to Y$. Show that $\mathcal{L} \otimes \mathcal{M}$ is very ample. (Hint: Reduce to the case where the target is affine. $\mathcal{L}$ induces a map to $\mathbb{P}^n_A$, and this corresponds to $n+1$ sections $s_0, \dots, s_n$ of $\mathcal{L}$. We also have a finite number m of sections $t_1, \dots, t_m$ of $\mathcal{M}$ which generate the stalks. Consider the $(n+1)m$ sections of $\mathcal{L} \otimes \mathcal{M}$ given by $s_i t_j$. Show that these sections are base-point-free, and hence induce a morphism to $\mathbb{P}^{(n+1)m-1}$. Show that it is a closed immersion.)

5. Suppose $\pi : X \to Y$ is proper and Y is quasicompact. Show that if $\mathcal{L}$ is relatively ample on X, then some tensor power of $\mathcal{L}$ is very ample.

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6. State and prove Serre's criterion for relative ampleness (where the target is quasicompact) by adapting the statement of Serre's criterion for ampleness. **Whoops! Ziyu and Rob point out that I used Serre's criterion as the *definition* of ampleness (and similarly, relative ampleness). Thus this exercise is nonsense.**

7. Use Serre's criterion for ampleness to prove that the pullback of ample sheaf on a projective scheme by a finite morphism is ample. Hence if a base-point-free invertible sheaf on a proper scheme induces a morphism to projective space that is finite onto its image, then it is is ample.

8. In class, we proved the following: Suppose $\pi : X \to \operatorname{Spec} B$ is proper, $\mathcal{L}$ ample, and $\mathcal{M}$ invertible. Then $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is very ample for $n \gg 0$. Give and prove the corresponding statement for a relatively ample invertible sheaf over a quasicompact base.

9. Suppose X a projective k-scheme. Show that every invertible sheaf is the difference of two *effective* Cartier divisors. Thus the groupification of the semigroup of effective Cartier divisors is the Picard group. Hence if you want to prove something about Cartier divisors on such a thing, you can study effective Cartier divisors. (This is false if projective is replaced by proper — ask Sam Payne for an example.)

10. Suppose C is a generically reduced projective k-curve. Then we can define degree of an invertible sheaf $\mathcal{M}$ as follows. Show that $\mathcal{M}$ has a meromorphic section that is regular at every singular point of C. Thus our old definition (number of zeros minus number of poles, using facts about discrete valuation rings) applies. Prove the Riemann-Roch theorem for generically reduced projective curves. (Hint: our original proof essentially will carry through without change.)

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