FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 17

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This set is due Thursday, April 20. You can hand it in to Rob Easton, in class or via his mailbox. It covers (roughly) classes 37 and 38.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 37:

1+. In class I stated the following. Note that if A is generated over B (as an algebra) by $x_i \in A$ (where i lies in some index set, possibly infinite), subject to some relations r_j (where j lies in some index set, and each is a polynomial in some finite number of the x_i), then the A-module $\Omega_{A/B}$ is generated by the dx_i, subject to the relations (i)—(iii) and $dr_j = 0$. In short, we needn't take every single element of A; we can take a generating set. And we needn't take every single relation among these generating elements; we can take generators of the relations. Verify this.

2. *(localization of differentials)* If S is a multiplicative set of A, show that there is a natural isomorphism $\Omega_{S^{-1}A/B} \cong S^{-1}\Omega_{A/B}$. (Again, this should be believable from the intuitive picture of "vertical cotangent vectors".) If T is a multiplicative set of B, show that there is a natural isomorphism $\Omega_{S^{-1}A/T^{-1}B} \cong S^{-1}\Omega_{A/B}$ where S is the multiplicative set of A that is the image of the multiplicative set $T \subset B$.

3+. (a) *(pullback of differentials)* If

$$\begin{array}{ccc} A' & \longleftarrow & A \\ \uparrow & & \uparrow \\ B' & \longleftarrow & B \end{array}$$

is a commutative diagram, show that there is a natural homomorphism of A'-modules $\Omega_{A/B} \otimes_A A' \to \Omega_{A'/B'}$. An important special case is $B = B'$.
(b) *(differentials behave well with respect to base extension, affine case)* If furthermore the above diagram is a tensor diagram (i.e. $A' \cong B' \otimes_B A$) then show that $\Omega_{A/B} \otimes_A A' \to \Omega_{A'/B'}$ is an isomorphism.

4. Suppose k is a field, and K is a separable algebraic extension of k. Show that $\Omega_{K/k} = 0$.

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5. *(Jacobian description of $\Omega_{A/B}$)* Suppose $A = B[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Then $\Omega_{A/B} = \{\oplus_i B dx_i\}/\{df_j = 0\}$ maybe interpreted as the cokernel of the Jacobian matrix $J : A^{\oplus r} \to A^{\oplus n}$.

Class 38:

6. *(normal bundles to effective Cartier divisors)* Suppose $D \subset X$ is an effective Cartier divisor. Show that the conormal sheaf $\mathcal{N}^{\vee}_{D/X}$ is $\mathcal{O}(-D)|_D$ (and in particular is an invertible sheaf), and hence that the normal sheaf is $\mathcal{O}(D)|_D$. It may be surprising that the normal sheaf should be locally free if $X \cong \mathbb{A}^2$ and D is the union of the two axes (and more generally if X is nonsingular but D is singular), because you may be used to thinking that the normal bundle is isomorphic to a "tubular neighborhood".

7-. Suppose $f : X \to Y$ is locally of finite type, and X is locally Noetherian. Show that $\Omega_{X/Y}$ is a coherent sheaf on X.

8+. *(differentials on hyperelliptic curves)* Consider the double cover $f : C \to \mathbb{P}^1_k$ branched over $2g + 2$ distinct points. (We saw earlier that this curve has genus g.) Then $\Omega_{C/k}$ is again an invertible sheaf. What is its degree? (Hint: let x be a coordinate on one of the coordinate patches of $\mathbb{P}^1_k$. Consider $f^* dx$ on C, and count poles and zeros.) In class I gave a sketch showing that you should expect the answer to be $2g - 2$.

9. *(differentials on non-singular plane curves)* Suppose C is a nonsingular plane curve of degree d in $\mathbb{P}^2_k$, where k is algebraically closed. By considering coordinate patches, find the degree of $\Omega_{C/k}$. Make any reasonable simplifying assumption (so that you believe that your result still holds for "most" curves).

10. Suppose that C is a nonsingular projective curve over k such that $\Omega_{C/k}$ is an invertible sheaf. (We'll see that for nonsingular curves, the sheaf of differentials is always locally free. But we don't yet know that.) Let $C_{\overline{k}} = C \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. Show that $\Omega_{C_{\overline{k}}/\overline{k}}$ is locally free, and that

$$\deg \Omega_{C_{\overline{k}}/\overline{k}} = \deg \Omega_{C/k}.$$

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