FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 15

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This set is due Thursday, March 9, in Jarod Alper's mailbox. It covers (roughly) classes 33 and 34.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 33:

1. *(for people who like non-algebraically closed fields)* Suppose that X is a quasicompact separated k-scheme, where k is a field. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, and $f : X_{\overline{k}} \to X$ the projection. Describe a natural isomorphism $H^i(X, \mathcal{F}) \otimes_k \overline{k} \to H^i(X_{\overline{k}}, f^*\mathcal{F})$. Recall that a k-scheme X is *geometrically integral* if $X_{\overline{k}}$ is integral. Show that if X is geometrically integral, then $H^0(X, \mathcal{O}_X) \cong k$. (This is a clue that $\mathbb{P}^1_{\mathbb{C}}$ is not a geometrically integral $\mathbb{R}$-scheme.)

2. Suppose Y is any scheme, and $\pi : \mathbb{P}^n_Y \to Y$ is the trivial projective bundle over Y. Show that $\pi_* \mathcal{O}_{\mathbb{P}^n_Y} \cong \mathcal{O}_Y$. More generally, show that $R^j \pi_* \mathcal{O}(m)$ is a finite rank free sheaf on Y, and is 0 if $j \neq 0, n$. Find the rank otherwise.

3. Let A be any ring. Suppose a is a negative integer and b is a positive integer. Show that $H^i(\mathbb{P}^m_A \times_A \mathbb{P}^n_A, \mathcal{O}(a, b))$ is 0 unless $i = m$, in which case it is a free A-module. Find the rank of this free A-module. (Hint: Use the previous exercise, and the projection formula, which was Exercise 1.3 of class 32, and exercise 17 of problem set 14.)

4. (a) Find the genus of a curve in class $(2, n)$ on $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$. (A curve in class $(2, n)$ is any effective Cartier divisor corresponding to invertible sheaf $\mathcal{O}(2, n)$. Equivalently, it is a curve whose ideal sheaf is isomorphic to $\mathcal{O}(-2, -n)$. Equivalently, it is a curve cut out by a non-zero form of bidegree $(2, n)$.)
(b) Suppose for convenience that k is algebraically closed of characteristic not 2. Show that there exists an integral nonsingular curve in class $(2, n)$ on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ for each $n > 0$.

5. Suppose X and Y are projective k-schemes, and $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on X and Y respectively. Recall that if $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the two projections, then $\mathcal{F} \boxtimes \mathcal{G} := \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{G}$. Prove the following, adding additional hypotheses if you find

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them necessary.
(a) Show that $H^0(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^0(X, \mathcal{F}) \otimes H^0(Y, \mathcal{G})$.
(b) Show that $H^{\dim X + \dim Y}(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^{\dim X}(X, \mathcal{F}) \otimes_k H^{\dim Y}(Y, \mathcal{G})$.
(c) Show that $\chi(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \chi(X, \mathcal{F})\chi(Y, \mathcal{G})$.

Class 34:

6-. Show that the following two morphisms are projective morphisms that are injective on points, but that are not injective on tangent vectors.
(a) the normalization of the cusp $y^2 = x^3$ in the plane
(b) the Frobenius morphism from $\mathbb{A}^1$ to $\mathbb{A}^1$, given by $k[t] \to k[u]$, $u \to t^p$, where k has characteristic p.

7. Suppose $\mathcal{L}$ is a degree $2g - 2$ invertible sheaf. Show that it has $g - 1$ or g sections, and it has g sections if and only if $\mathcal{L} \cong \mathcal{K}$.

8. Suppose C is a genus 0 curve (projective, geometrically integral and nonsingular). Show that C has a point of degree at most 2.

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