FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 14

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This set is due Thursday, March 2, in Jarod Alper's mailbox. It covers (roughly) classes 31 and 32.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 31:

1-. Prove the base case of Theorem 1.1 of Class 31. If you choose to do the case $k = -1$, explain precisely why what you are proving is the base case!

2-. Consider the short exact sequence of A-modules $0 \longrightarrow M \xrightarrow{\times f} M \longrightarrow K' \longrightarrow 0$. Show that $\operatorname{Supp} K' = \operatorname{Supp}(M) \cap \operatorname{Supp}(f)$.

3-. Show that the twisted cubic (in $\mathbb{P}^3$) has Hilbert polynomial $3m + 1$.

4. (a) Find the Hilbert polynomial for the dth Veronese embedding of $\mathbb{P}^n$ (i.e. the closed immersion of $\mathbb{P}^n$ in a bigger projective space by way of the line bundle $\mathcal{O}(d)$)
(b) Find the degree of the dth Veronese embedding of $\mathbb{P}^n$.

5-. Show that the degree of a degree d hypersurface is d (preventing a notational crisis).

6. Suppose a curve C is embedded in projective space via an invertible sheaf of degree d. (In other words, this line bundle determines a closed immersion.) Show that the degree of C under this embedding is d (preventing another notational crisis). (Hint: Riemann-Roch.)

7+. *(Bezout's theorem)* Suppose X is a projective scheme of dimension at least 1, and H is a degree d hypersurface not containing any associated points of X. (For example, if X is a projective variety, then we are just requiring H not to contain any irreducible components of X.) Show that $\deg H \cap X = d \deg X$.

8-. Determine the degree of the d-fold Veronese embedding of $\mathbb{P}^n$ in a different way as follows. Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ be the Veronese embedding. To find the degree of the image,

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we intersect it with n hyperplanes in $\mathbb{P}^N$ (scheme-theoretically), and find the number of intersection points (counted with multiplicity). But the pullback of a hyperplane in $\mathbb{P}^N$ to $\mathbb{P}^n$ is a degree d hypersurface. Perform this intersection in $\mathbb{P}^n$, and use Bezout's theorem. (If already you know the answer by the earlier exercise on the degree of the Veronese embedding, this will be easier.)

9+. Show that if X is a complete intersection of dimension r in $\mathbb{P}^n$, then $H^i(X, \mathcal{O}_X(m)) = 0$ for all $0 < i < r$ and all m. Show that if $r > 0$, then $H^0(\mathbb{P}^n, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m))$ is surjective.

10-. Show that complete intersections of positive dimension are connected. (Hint: show $h^0(X, \mathcal{O}_X) = 1$.)

11-. Find the genus of the intersection of 2 quadrics in $\mathbb{P}^3$. (We get curves of more genera by generalizing this!)

12-. Show that the rational normal curve of degree d in $\mathbb{P}^d$ is *not* a complete intersection if $d > 2$.

13-. Show that the union of 2 distinct planes in $\mathbb{P}^4$ is not a complete intersection. (This is the first appearance of another universal counterexample!) Hint: it is connected, but you can slice with another plane and get something not connected.

14. Show that if π is affine, then for $i > 0$, $R^i\pi_*\mathcal{F} = 0$. Moreover, if Y is quasicompact and separated, show that the natural morphism $H^i(X, \mathcal{F}) \to H^i(Y, f_*\mathcal{F})$ is an isomorphism. (A special case of the first sentence is a special case we showed earlier, when π is a closed immersion. Hint: use any affine cover on Y, which will induce an affine cover of X.)

Class 32:

15+. *(Important algebra exercise)* Suppose $M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3$ is a complex of A-modules (i.e. $\beta \circ \alpha = 0$), and N is an A-module. (a) Describe a natural homomorphism of the cohomology of the complex, tensored with N, with the cohomology of the complex you get when you tensor with N $H(M_*) \otimes_A B \to H(M_* \otimes_A N)$, i.e.

$$\left(\frac{\ker \beta}{\operatorname{im} \alpha}\right) \otimes_A N \to \frac{\ker(\beta \otimes N)}{\operatorname{im}(\alpha \otimes N)}.$$

I always forget which way this map is supposed to go.
(b) If N is *flat*, i.e. $\otimes N$ is an exact functor, show that the morphism defined above is an isomorphism. (Hint: This is actually a categorical question: if M_* is an exact sequence in an abelian category, and F is a right-exact functor, then (a) there is a natural morphism $FH(M_*) \to H(FM_*)$, and (b) if F is an exact functor, this morphism is an isomorphism.)

16+. *(Higher pushforwards and base change)* (a) Suppose $f : Z \to Y$ is any morphism, and $\pi : X \to Y$ as usual is quasicompact and separated. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let

$$\begin{CD} W @>{f'}>> X \\ @V{\pi'}VV @VV{\pi}V \\ Z @>{f}>> Y \end{CD}$$

is a fiber diagram. Describe a natural morphism $f^*(R^i\pi_*\mathcal{F}) \to R^i\pi'_*(f')^*\mathcal{F}$.
(b) If $f : Z \to Y$ is an affine morphism, and for a cover $\operatorname{Spec} A_i$ of Y, where $f^{-1}(\operatorname{Spec} A_i) = \operatorname{Spec} B_i$, B_i is a *flat* A-algebra, show that the natural morphism of (a) is an isomorphism. (You can likely generalize this immediately, but this will lead us into the concept of flat morphisms, and we'll hold off discussing this notion for a while.)

17+. *(The projection formula)* Suppose $\pi : X \to Y$ is quasicompact and separated, and $\mathcal{E}$, $\mathcal{F}$ are quasicoherent sheaves on X and Y respectively. (a) Describe a natural morphism

$$(R^i\pi_*\mathcal{E}) \otimes \mathcal{F} \to R^i\pi_*(\mathcal{E} \otimes \pi^*\mathcal{F}).$$

(b) If $\mathcal{F}$ is locally free, show that this natural morphism is an isomorphism.

18. Consider the open immersion $\pi : \mathbb{A}^n - 0 \to \mathbb{A}^n$. By direct calculation, show that $R^{n-1}f_*\mathcal{O}_{\mathbb{A}^n-0} \neq 0$.

19+. *(Semicontinuity of fiber dimension of projective morphisms)* Suppose $\pi : X \to Y$ is a projective morphism where $\mathcal{O}_Y$ is coherent. Show that $\{y \in Y : \dim f^{-1}(y) > k\}$ is a Zariski-closed subset. In other words, the dimension of the fiber "jumps over Zariski-closed subsets". (You can interpret the case $k = -1$ as the fact that projective morphisms are closed.) This exercise is rather important for having a sense of how projective morphisms behave! (Hint: see the notes.)

20. Suppose $f : X \to Y$ is a projective morphism, with $\mathcal{O}(1)$ on X. Suppose Y is quasicompact and $\mathcal{O}_Y$ is coherent. Let $\mathcal{F}$ be coherent on X. Show that

(a) $f^*f_*\mathcal{F}(n) \to \mathcal{F}(n)$ is surjective for $n \gg 0$. (First show that there is a natural map for any n! Hint: by adjointness of f_* with f_*.) Translation: for $n \gg 0$, $\mathcal{F}(n)$ is relatively generated by global sections.
(b) For $i > 0$ and $n \gg 0$, $R^if_*\mathcal{F}(n) = 0$.

21-. Show that $H^0(A^*) = E_\infty^{0,0} = E_2^{0,0}$ and

$$0 \to E_2^{1,0} \to H^1(A^*) \to E_2^{0,1} \to E_2^{2,0} \to H^2(A^*).$$

(Here take the spectral sequence starting with the vertical arrows.)

22. Suppose we are working in the category of vector spaces over a field k, and $\oplus_{p,q}E_2^{p,q}$ is a finite-dimensional vector space. Show that $\chi(H^*(A^*))$ is well-defined, and equals $\sum_{p,q}(-1)^{p+q}E_2^{p,q}$. (It will sometimes happen that $\oplus E_0^{p,q}$ will be an infinite-dimensional vector space, but that $E_2^{p,q}$ will be finite-dimensional!)

23. By looking at our spectral sequence proof of the five lemma, prove a subtler version of the five lemma, where one of the isomorphisms can instead just be required to be an injection, and another can instead just be required to be a surjection. (I'm deliberately not telling you which ones, so you can see how the spectral sequence is telling you how to improve the result.) I've heard this called the "subtle five lemma", but I like calling it the $4\frac{1}{2}$-lemma.

24. If β and δ (in (1)) are injective, and α is surjective, show that γ is injective. State the dual statement. (The proof of the dual statement will be essentially the same.)

$$\begin{tikzcd} F \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & J \\ A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & D \arrow[r] \arrow[u, "\delta"] & E \arrow[u, "\epsilon"] \end{tikzcd} \tag{1}$$

25. Use spectral sequences to show that a short exact sequence of complexes gives a long exact sequence in cohomology.

26. Suppose $\mu : A^* \to B^*$ is a morphism of complexes. Suppose C^* is the single complex associated to the double complex $A^* \to B^*$. (C^* is called the *mapping cone* of μ.) Show that there is a long exact sequence of complexes:

$$\cdots \to H^{i-1}(C^*) \to H^i(A^*) \to H^i(B^*) \to H^i(C^*) \to H^{i+1}(A^*) \to \cdots .$$

(There is a slight notational ambiguity here; depending on how you index your double complex, your long exact sequence might look slightly different.) In particular, people often use the fact μ induces an isomorphism on cohomology if and only if the mapping cone is exact.

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