FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 13

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This set is due Thursday, February 23, in Jarod Alper's mailbox. It covers (roughly) classes 29 and 30.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 29:

1+. (This was discussed in class 29, but I've put it in the class 27 notes, because it belongs more naturally there.) Suppose $W \hookrightarrow Y$ is a locally closed immersion. The scheme-theoretic closure is the smallest closed subscheme of Y containing W. Show that this notion is well-defined. More generally, if $f : W \to Y$ is any morphism, define the scheme-theoretic image as the smallest closed subscheme $Z \to Y$ so that f factors through $Z \hookrightarrow Y$. Show that this is well-defined. (One possible hint: use a universal property argument.) If Y is affine, the ideal sheaf corresponds to the functions on Y that are zero when pulled back to Z. Show that the closed set underlying the image subscheme may be strictly larger than the closure of the set-theoretic image: consider $\coprod_{n \geq 0} \operatorname{Spec} k[t]/t^n \to \operatorname{Spec} k[t]$. (I suspect that such a pathology cannot occur for finite type morphisms of Noetherian schemes, but I haven't investigated.)

2. Suppose $f : C \to C'$ is a degree d morphism of integral projective nonsingular curves, and $\mathcal{L}$ is an invertible sheaf on C'. Show that $\deg_C f^*\mathcal{L} = d \deg_{C'} \mathcal{L}$.

3. (for those who like working with non-Noetherian schemes) Suppose R is a ring that is coherent over itself. (In other words, R is a coherent R-module.) Show that for any coherent sheaf $\mathcal{F}$ on a projective R-scheme where R is Noetherian, $h^i(X, \mathcal{F})$ is a finitely generated R-module. (Hint: induct downwards as before. The order is as follows: $H^n(\mathbb{P}^n_R, \mathcal{F})$ finitely generated, $H^n(\mathbb{P}^n_R, \mathcal{G})$ finitely generated, $H^n(\mathbb{P}^n_R, \mathcal{F})$ coherent, $H^n(\mathbb{P}^n_R, \mathcal{G})$ coherent, $H^{n-1}(\mathbb{P}^n_R, \mathcal{F})$ finitely generated, $H^{n-1}(\mathbb{P}^n_R, \mathcal{G})$ finitely generated, etc.)

4+ (This is important!) Suppose $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of sheaves on a topological space, and $\mathcal{U}$ is an open cover such that on any intersection the sections of $\mathcal{F}_2$ surject onto $\mathcal{F}_3$. Show that we get a long exact sequence of cohomology. (Note that this applies in our case!)

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5. If D is an effective Cartier divisor on a projective nonsingular curve, say $D = \sum n_i p_i$, prove that $\deg D = \sum n_i \deg p_i$, where $\deg p_i$ is the degree of the field extension of the residue field at p_i over k.

Class 30:

6. Suppose $V \subset U$ are open subsets of X. Show that we have restriction morphisms $H^i(U, \mathcal{F}) \to H^i(V, \mathcal{F})$ (if U and V are quasicompact, and U hence V is separated). Show that restrictions commute. Hence if X is a Noetherian space, $H^i(;\mathcal{F})$ this is a contravariant functor from the category $\operatorname{Top}(X)$ to abelian groups. (The same argument will show more generally that for any map $f : X \to Y$, there exist natural maps $H^i(X, \mathcal{F}) \to H^i(X, f^*\mathcal{F})$; I should have asked this instead.)

7. Show that if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on separated and quasicompact X then we have natural maps $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$. Hence $H^i(X, \cdot)$ is a covariant functor from quasicoherent sheaves on X to abelian groups (or even R-modules).

8. Verify that $H^{n-1}(\mathbb{P}^{n-1}_R, \mathcal{F}') \to H^n(\mathbb{P}^n_R, \mathcal{F})$ is injective. (Hint: one possibility is by verifying that it is the map on Laurent monomials we claimed when proving that cohomology of $\mathcal{O}(m)$ is what we wanted it to be. In particular, this fact was used in that proof, so you can't use that theorem!)

9. Suppose X is a projective k-scheme. Show that Euler characteristic is additive in exact sequences. In other words, if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on X, then $\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$. (Hint: consider the long exact sequence in cohomology.) More generally, if

$$0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$$

is an exact sequence of sheaves, show that

$$\sum_{i=1}^{n} (-1)^i \chi(X, \mathcal{F}_i) = 0.$$

10. *The Riemann-Roch theorem for line bundles on nonsingular projective curves over* k. Suppose $\mathcal{F}$ is an invertible sheaf on C. Show that $\chi(\mathcal{L}) = \deg \mathcal{L} + \chi(C, \mathcal{O}_C)$. (Possible hint: Write $\mathcal{L}$ as the difference of two effective Cartier divisors, $\mathcal{L} \cong \mathcal{O}(Z - P)$. Describe two exact sequences $0 \to \mathcal{L}(-Z) \to \mathcal{L} \to \mathcal{O}_Z \otimes \mathcal{L} \to 0$ and $0 \to \mathcal{O}_C(-P) \to \mathcal{O}_C \to \mathcal{O}_P \to 0$, where $\mathcal{L}(-Z) \cong \mathcal{O}_C(P)$.)

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