FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 12

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This set is due Thursday, February 16, in Jarod Alper's mailbox. It covers (roughly) classes 27 and 28.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 27:

1+. (Scheme-theoretic closure and scheme-theoretic image) If $f : W \to Y$ is any morphism, define the scheme-theoretic image as the smallest closed subscheme $Z \to Y$ so that f factors through $Z \hookrightarrow Y$. Show that this is well-defined. (One possible hint: use a universal property argument.) If Y is affine, the ideal sheaf corresponds to the functions on Y that are zero when pulled back to Z. Show that the closed set underlying the image subscheme may be strictly larger than the closure of the set-theoretic image: consider $\operatorname{Spec} k(t) \to \operatorname{Spec} k[t]$. (We define the scheme-theoretic closure of a locally closed subscheme $W \hookrightarrow Y$ as the scheme-theoretic image of the morphism.)

2-. Show that rational functions on an integral scheme correspond to rational maps to $\mathbb{A}^1_{\mathbb{Z}}$.

3-. Show that you can compose two rational maps $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$ if f is dominant.

4. We define the *graph* of a rational map $f : X \dashrightarrow Y$ as follows: let (U, f') be any representative of this rational map (so $f' : U \to Y$ is a morphism). Let Γ_f be the scheme-theoretic closure of $\Gamma_{f'} \hookrightarrow U \times Y \hookrightarrow X \times Y$, where the first map is a closed immersion, and the second is an open immersion. Show that this is independent of the choice of U.

5. Let K be a finitely generated field extension of transcendence degree m over k. Show there exists an irreducible k-variety W with function field K. (Hint: let $x_1, \ldots, x_n$ be generators for K over k. Consider the map $\operatorname{Spec} K \to \operatorname{Spec} k[t_1, \ldots, t_n]$ given by the ring map $t_i \mapsto x_i$. Take the scheme-theoretic closure of the image.)

Date: Tuesday, February 7, 2006. Updated March 8.

6+. Prove the following. Suppose X and Y are integral and separated (our standard hypotheses from last day). Then X and Y are birational if and only if there is a dense=non-empty open subscheme U of X and a dense=non-empty open subscheme V of Y such that $U \cong Y$. (Feel free to consult Iitaka, or Hartshorne Chapter I Corollary 4.5.)

7. Use the class discussion to find a "formula" for all Pythagorean triples.

8. Show that the conic $x^2+y^2 = z^2$ in $\mathbb{P}^2_k$ is isomorphic to $\mathbb{P}^1_k$ for any field k of characteristic not 2. (Presumably this is true for any ring in which 2 is invertible too...)

9. Find all rational solutions to the $y^2 = x^3 + x^2$, by finding a birational map to $\mathbb{A}^1$, mimicking what worked with the conic.

10. Find a birational map from the quadric $Q = \{x^2 + y^2 = w^2 + z^2\}$ to $\mathbb{P}^2$. Use this to find all rational points on Q. (This illustrates a good way of solving Diophantine equations. You will find a dense open subset of Q that is isomorphic to a dense open subset of $\mathbb{P}^2$, where you can easily find all the rational points. There will be a closed subset of Q where the rational map is not defined, or not an isomorphism, but you can deal with this subset in an ad hoc fashion.)

11. *(a first view of a blow-up)* Let k be an algebraically closed field. (We make this hypothesis in order to not need any fancy facts on nonsingularity.) Consider the rational map $\mathbb{A}^2_k \dashrightarrow \mathbb{P}^1_k$ given by $(x, y) \mapsto [x; y]$. I think you have shown earlier that this rational map cannot be extended over the origin. Consider the graph of the birational map, which we denote $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$. It is a subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$. Show that if the coordinates on $\mathbb{A}^2$ are x, y, and the coordinates on $\mathbb{P}^1$ are u, v, this subscheme is cut out in $\mathbb{A}^2 \times \mathbb{P}^1$ by the single equation $xv = yu$. Show that $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$ is nonsingular. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{P}^1_k$ over each closed point of $\mathbb{P}^1_k$. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{A}^2_k$. Show that the fiber over $(0, 0)$ is an effective Cartier divisor. It is called the *exceptional divisor*.

12. *(the Cremona transformation, a useful classical construction)* Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by $[x; y; z] \to [1/x; 1/y; 1/z]$. What is the the domain of definition? (It is bigger than the locus where $xyz \neq 0$!) You will observe that you can extend it over codimension 1 sets. This will again foreshadow a result we will soon prove.

Class 28:

13. *(Useful practice!)* Suppose X is a Noetherian k-scheme, and Z is an irreducible codimension 1 subvariety whose generic point is a nonsingular point of X (so the local ring $\mathcal{O}_{X,Z}$ is a discrete valuation ring). Suppose $X \dashrightarrow Y$ is a rational map to a projective k-scheme. Show that the domain of definition of the rational map includes a dense open subset of Z. In other words, rational maps from Noetherian k-schemes to projective k-schemes can be extended over nonsingular codimension 1 sets. See problem 12 to see this principle in action. (By the easy direction of the valuative criterion of separatedness, or the theorem of uniqueness of extensions of maps from reduced schemes to separated schemes — Theorem 3.3 of Class 27 — this map is unique.)

14. Show that all nonsingular proper curves are projective. (We may eventually see that all reduced proper curves over k are projective, but I'm not sure; this will use the Riemann-Roch theorem, and I may just prove it for projective curves.)

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