FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 11

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This set is due Thursday, February 9, in Jarod Alper's mailbox. It covers (roughly) classes 25 and 26.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 25:

1. Verify that the following definition of "induced reduced subscheme structure" is well-defined. Suppose X is a scheme, and S is a *closed subset* of X. Then there is a unique reduced closed subscheme Z of X "supported on S". More precisely, it can be defined by the following universal property: for any morphism from a *reduced* scheme Y to X, whose image lies in S (as a set), this morphism factors through Z uniquely. Over an affine $X = \operatorname{Spec} R$, we get $\operatorname{Spec} R/I(S)$. (For example, if S is the entire underlying set of X, we get X^{red}.)

2+. Show that open immersions and closed immersions are separated. (Answer: Show that monomorphisms are separated. Open and closed immersions are monomorphisms, by earlier exercises. Alternatively, show this by hand.)

3+. Show that every morphism of affine schemes is separated. (Hint: this was essentially done in the notes if you know where to look.)

4. Complete the proof that $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is separated, by verifying the last sentence in the proof.

5. Show that the line with doubled origin X is not separated, by verifying that the image of the diagonal morphism is not closed. (Another argument is given below, in Exercise 12.)

6. Show that any morphism from a Noetherian scheme is quasicompact. Hence show that any morphism from a Noetherian scheme is quasiseparated.

Date: Tuesday, January 31, 2006. Update February 27 (changes to 18). Update February 12 (change to 15).

7+. Show that $f : X \to Y$ is quasiseparated if and only if for any affine open $\operatorname{Spec} R$ of Y, and two affine open subsets U and V of X mapping to $\operatorname{Spec} R$, $U \cap V$ is a *finite* union of affine open sets.

8. Here is an example of a nonquasiseparated scheme. Let $X = \operatorname{Spec} k[x_1, x_2, \dots]$, and let U be $X - \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, \dots)$. Take two copies of X, glued along U. Show that the result is not quasiseparated.

9. Prove that the condition of being quasiseparated is local on the target. (Hint: the condition of being quasicompact is local on the target by an earlier exercise.)

10-. Show that a k-scheme is separated (over k) iff it is separated over $\mathbb{Z}$.

11+ (the locus where two morphisms agree) We can now make sense of the following statement. Suppose

$$f, g : \quad \begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

are two morphisms over Z. Then the locus on X where f and g agree is a locally closed subscheme of X. If $Y \to Z$ is separated, then the locus is a closed subscheme of X. More precisely, define V to be the following fibered product:

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d, "\delta"] \\ X \arrow[r, "(f,g)"] & Y \times_Z Y. \end{tikzcd}$$

As δ is a locally closed immersion, $V \to X$ is too. Then if $h : W \to X$ is any scheme such that $g \circ h = f \circ h$, then h factors through V. (Put differently: we are describing $V \hookrightarrow X$ by way of a universal property. Taking this as the definition, it is not a priori clear that V is a locally closed subscheme of X, or even that it exists.) Now we come to the exercise: prove this (the sentence before the parentheses). (Hint: we get a map $g \circ h = f \circ h : W \to Y$. Use the definition of fibered product to get $W \to V$.)

12. Show that the line with doubled origin X is not separated, by finding two morphisms $f_1, f_2 : W \to X$ whose domain of agreement is not a closed subscheme. (Another argument was given above, in Exercise 5.)

13. Suppose $\pi : Y \to X$ is a morphism, and $s : X \to Y$ is a *section* of a morphism, i.e. $\pi \circ s$ is the identity on X. Show that s is a locally closed immersion. Show that if π is separated, then s is a closed immersion. (This generalizes Proposition 1.12 in the Class 25 notes.)

14-. Suppose P is a class of morphisms such that closed immersions are in P, and P is closed under fibered product and composition. Show that if $X \to Y$ is in P then $X^{\text{red}} \to Y^{\text{red}}$ is in P. (Two examples are the classes of separated morphisms and quasiseparated morphisms.) (Hint:

$$\begin{tikzcd} X^{red} \arrow[r] \arrow[dr] & X \times_Y Y^{red} \arrow[r] \arrow[d] & Y^{red} \arrow[d] \\ & X \arrow[r] & Y \end{tikzcd}$$

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15. Suppose $\pi : X \to Y$ is a morphism or a ring R, Y is a separated R-scheme, U is an affine open subset of X, and V is an affine open subset of Y. Show that $U \cap \pi^{-1}V$ is an affine open subset of X. (Hint: this generalizes Proposition 1.9 of the Class 25 notes. Use Proposition 1.12 or 1.13.) This will be used in the proof of the Leray spectral sequence.

16. Make this precise: show that the line with the doubled origin fails the valuative criterion for separatedness.

Class 26:

17-. Show that $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{C}$ is not proper.

18. Show that finite morphisms are projective. (There was something that I didn't check in the notes.) More explicitly, if $X \to Y$ is finite, then I described a sheaf of graded algebras $\mathcal{S}_*$ on Y, and claimed that $X = \underline{\mathrm{Proj}}\mathcal{S}_*$. Verify that this is indeed the case. What is $\mathcal{O}_{\underline{\mathrm{Proj}}\mathcal{S}_*}(1)$?

19-. Suppose (1) is a commutative diagram, and f is surjective, g is proper, and h is separated and finite type. Show that h is proper.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd} \tag{1}$$

(I'm not sure that this is useful, but I know that if I forget to mention it, it will come back to haunt me!)

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