FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 10

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This set is due Thursday, February 2, in Jarod Alper's mailbox. It covers (roughly) classes 23 and 24.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

0. Here is something I would like to see worked out. Show that the points of $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ are in natural bijection with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the Zariski topology on the former agrees with the profinite topology on the latter.

Class 23:

1-. Show that for the morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$, all geometric fibers consist of two reduced points.

2+. Show that the notion of "morphism locally of finite type" is preserved by base change. Show that the notion of "affine morphism" is preserved by base change. Show that the notion of "finite morphism" is preserved by base change.

3+. Show that the notion of "morphism of finite type" is preserved by base change.

4. Show that the notion of "quasicompact morphism" is preserved by base change.

5. Show that the notion of "quasifinite morphism" (= finite type + finite fibers) is preserved by base change. (Note: the notion of "finite fibers" is not preserved by base change. $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ has finite fibers, but $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \operatorname{Spec} \overline{\mathbb{Q}}$ has one point for each element of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.)

6. Show that surjectivity is preserved by base change (or fibered product). In other words, if $X \to Y$ is a surjective morphism, then for any $Z \to Y$, $X \times_Y Z \to Z$ is surjective. (You may end up using the fact that for any fields k_1 and k_2 containing k_3, $k_1 \otimes_{k_3} k_2$ is non-zero, and also the axiom of choice.)

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7-. Show that the notion of "irreducible" is not necessarily preserved by base change. Show that the notion of "connected" is not necessarily preserved by base change. (Hint: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{Q}[i] \otimes_{\mathbb{Q}} \mathbb{Q}[i]$.)

8. Show that $\operatorname{Spec} \mathbb{C}$ is not a geometrically irreducible $\mathbb{R}$-scheme. If $\operatorname{char} k = p$, show that $\operatorname{Spec} k(u)$ is not a geometrically reduced $\operatorname{Spec} k(u^p)$-scheme.

9. Show that the notion of geometrically irreducible (resp. connected, reduced, integral) fibers behaves well with respect to base change.

10. Suppose that l/k is a finite field extension. Show that a k-scheme X is normal if and only if $X \times_{\operatorname{Spec} k} \operatorname{Spec} l$ is normal. Hence deduce that if k is any field, then $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ is normal. Hint: we showed earlier (Problem B4 on set 4) that $\operatorname{Spec} k[a, b, c, d]/(a^2 + b^2 + c^2 + d^2)$ is normal. (This is less important, but helps us understand this example.)

11. Show that the Segre scheme (the image of the Segre morphism) is cut out by the equations corresponding to

$$\operatorname{rank} \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{m0} & \cdots & a_{mn} \end{pmatrix} = 1,$$

i.e. that all 2×2 minors vanish. (Hint: suppose you have a polynomial in the a_{ij} that becomes zero upon the substitution $a_{ij} = x_i y_j$. Give a recipe for subtracting polynomials of the form monomial times 2×2 minor so that the end result is 0.)

12. Show that $X^{\mathrm{red}} \to X$ satisfies the following universal property: any morphism from a reduced scheme Y to X factors uniquely through X^{red}.

$$\begin{array}{ccc} Y & \overset{\exists!}{\dashrightarrow} & X^{\mathrm{red}} \\ & \searrow \quad \swarrow & \\ & X & \end{array} .$$

(Do this exercise if you want to see how this sort of argument works in general.)

13. Show that $\nu : \operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ satisfies the universal property of normalization. We used this to show that normalization exists.

14. Show that normalizations exist for any quasiaffine X (i.e. any X that can be expressed as an open subset of an affine scheme). Show that normalizations exist in general.

Class 24:

15. Show that the normalization morphism is surjective. (Hint: Going-up!)

16. Show that $\dim \tilde{X} = \dim X$ (hint: see our going-up discussion).

17. Show that if X is an integral finite-type k-scheme, then its normalization $\nu : \tilde{X} \to X$ is a finite morphism.

18. Explain how to generalize the notion of normalization to the case where X is a reduced Noetherian scheme (with possibly more than one component). This basically requires defining a universal property. I'm not sure what the "perfect" definition, but all reasonable universal properties should lead to the same space.

19. Show that if X is an integral finite type k-scheme, then its non-normal points form a closed subset. (This is a bit trickier. Hint: consider where $\nu_* \mathcal{O}_{\tilde{X}}$ has rank 1.) I haven't thought through all the details recently, so I hope I've stated this correctly.

20. (Good practice with the concept.) Suppose $X = \operatorname{Spec} \mathbb{Z}[15i]$. Describe the normalization $\tilde{X} \to X$. (Hint: it isn't hard to find an integral extension of $\mathbb{Z}[15i]$ that is integrally closed. By the above discussion, you've then found the normalization!) Over what points of X is the normalization not an isomorphism?

21. (This is an important generalization!) Suppose X is an integral scheme. Define the *normalization of* X, $\nu : \tilde{X} \to X$, *in a given finite field extension of the function field of* X. Show that $\tilde{X}$ is normal. (This will be hard-wired into your definition.) Show that if either X is itself normal, or X is finite type over a field k, then the normalization in a finite field extension is a finite morphism.

22. Suppose $X = \operatorname{Spec} \mathbb{Z}$ (with function field $\mathbb{Q}$). Find its integral closure in the field extension $\mathbb{Q}(i)$.

23. (a) Suppose $X = \operatorname{Spec} k[x]$ (with function field $k(x)$). Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (We get a Dedekind domain.)
(b) Suppose $X = \mathbb{P}^1$, with distinguished open $\operatorname{Spec} k[x]$. Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (Part (a) involves computing the normalization over one affine open set; now figure out what happens over the "other".)

24. Show that if $f : Z \to X$ is an affine morphism, then we have a natural isomorphism $Z \cong \underline{\operatorname{Spec}}\, f_* \mathcal{O}_Z$ of X-schemes.

25. ($\underline{\operatorname{Spec}}$ behaves well with respect to base change) Suppose $f : Z \to X$ is any morphism, and $\mathcal{A}$ is a quasicoherent sheaf of algebras on X. Show that there is a natural isomorphism $Z \times_X \underline{\operatorname{Spec}}\, \mathcal{A} \cong \underline{\operatorname{Spec}}\, f^* \mathcal{A}$.

26. If $\mathcal{F}$ is a locally free sheaf, show that $\underline{\operatorname{Spec}} \operatorname{Sym} \mathcal{F}^*$ is a vector bundle, i.e. that given any point $p \in X$, there is a neighborhood $p \in U \subset X$ such that $\underline{\operatorname{Spec}} \operatorname{Sym} \mathcal{F}^*|_U \cong \mathbb{A}^1_U$. Show that $\mathcal{F}$ is a sheaf of sections of it.

27. Suppose $f : \underline{\operatorname{Spec}}\, \mathcal{A} \to X$ is a morphism. Show that the category of quasicoherent sheaves on $\underline{\operatorname{Spec}}\, A$ is "essentially the same" (=equivalent) as the category of quasicoherent sheaves on X with the structure of $\mathcal{A}$-modules (quasicoherent $\mathcal{A}$-modules on X).

28. Complete this argument that if $X = \operatorname{Spec} A$, then $(\operatorname{Proj} \mathcal{S}_*, \mathcal{O}(1))$ satisfies the universal property.

29. Show that $(\operatorname{Proj}\mathcal{S}_*, \mathcal{O}(1))$ exists in general, by following the analogous universal property argument: show that it exists for X quasiaffine, then in general.

30. ($\underline{\text{Proj}}$ behaves well with respect to base change) Suppose $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\text{Proj}}\mathcal{S}_*$ to exist. Let $f : Y \to X$ be any morphism. Give a natural isomorphism

$$(\underline{\text{Proj}} f^*\mathcal{S}_*, \mathcal{O}_{\underline{\text{Proj}} f^*\mathcal{S}_*}(1)) \cong (Y \times_X \underline{\text{Proj}}\mathcal{S}_*, g^*\mathcal{O}_{\underline{\text{Proj}}\mathcal{S}_*}(1)) \cong$$

where g is the natural morphism in the base change diagram

$$\begin{array}{ccc} Y \times_X \underline{\text{Proj}}\mathcal{S}_* & \xrightarrow{g} & \underline{\text{Proj}}\mathcal{S}_* \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

31. $\underline{\text{Proj}}(\mathcal{S}_*[t]) \cong \underline{\text{Spec}}\,\mathcal{S}_* \coprod \underline{\text{Proj}}\mathcal{S}_*$, where $\underline{\text{Spec}}\,\mathcal{S}_*$ is an open subscheme, and $\underline{\text{Proj}}\mathcal{S}_*$ is a closed subscheme. Show that $\underline{\text{Proj}}\mathcal{S}*$ is an effective Cartier divisor, corresponding to the invertible sheaf $\mathcal{O}_{\underline{\text{Proj}}N}(1)$. (This is the generalization of the projective and affine cone. At some point I should give an explicit reference to our earlier exercise on this.)

32. Suppose $\mathcal{L}$ is an invertible sheaf on X, and $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\text{Proj}}\mathcal{S}_*$ to exist. Define $\mathcal{S}'_* = \oplus_{n=0}\mathcal{S}_n \otimes \mathcal{L}_n$. Give a natural isomorphism of X-schemes

$$(\underline{\text{Proj}}\mathcal{S}'_*, \mathcal{O}_{\underline{\text{Proj}}\mathcal{S}'_*}(1)) \cong (\underline{\text{Proj}}\mathcal{S}_*, \mathcal{O}_{\underline{\text{Proj}}\mathcal{S}_*}(1) \otimes \pi^*\mathcal{L}),$$

where $\pi : \underline{\text{Proj}}\mathcal{S}_* \to X$ is the structure morphism. In other words, informally speaking, the $\underline{\text{Proj}}$ is the same, but the $\mathcal{O}(1)$ is twisted by $\mathcal{L}$.

33. Show that closed immersions are projective morphisms. (Hint: Suppose the closed immersion $X \to Y$ corresponds to $\mathcal{O}_Y \to \mathcal{O}_X$. Consider $\mathcal{S}_0 = \mathcal{O}_X$, $\mathcal{S}_i = \mathcal{O}_Y$ for $i > 1$.)

34. (suggested by Kirsten) Suppose $f : X \hookrightarrow \mathbb{P}^n_S$ where S is some scheme. Show that the structure morphism $\pi : X \to S$ is a projective morphism as follows: let $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n_S}(1)$, and show that $X = \underline{\text{Proj}}\pi_*\mathcal{L}^{\otimes n}$.

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