FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 9

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This set is due Tuesday, January 24, in Jarod Alper's mailbox. It covers (roughly) classes 19 through 22. (This is a long one, because I'm giving you the option of doing some problems from the end of last quarter.)

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in four solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 19:

1+. Show that morphisms $X \to \operatorname{Spec} A$ are in natural bijection with ring morphisms $A \to \Gamma(X, \mathcal{O}_X)$. (Hint: Show that this is true when X is affine. Use the fact that morphisms glue.)

2. Show that $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes. In other words, if X is any scheme, there exists a unique morphism to $\operatorname{Spec} \mathbb{Z}$. (Hence the category of schemes is isomorphic to the category of $\mathbb{Z}$-schemes.)

3. Show that morphisms $X \to \operatorname{Spec} \mathbb{Z}[t]$ correspond to global sections of the structure sheaf.

4. Show that global sections of $\mathcal{O}_X^*$ correspond naturally to maps to $\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$. ($\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ is a *group scheme*.)

5+. Suppose X is a finite type k-scheme. Describe a natural bijection $\operatorname{Hom}(\operatorname{Spec} k[\epsilon]/\epsilon^2, X)$ to the data of a k-valued point (a point whose residue field is k, necessarily closed) and a tangent vector at that point.

6. Suppose $i : U \to Z$ is an open immersion, and $f : Y \to Z$ is any morphism. Show that $U \times_Z Y$ exists. (Hint: I'll even tell you what it is: $(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)})$.)

7-. Show that open immersions are monomorphisms.

8+. Suppose $Y \to Z$ is a closed immersion, and $X \to Z$ is any morphism. Show that the fibered product $X \times_Y Z$ exists, by explicitly describing it. Show that the projection $X \times_Y Z \to Y$ is a closed immersion. We say that "closed immersions are preserved by

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base change" or "closed immersions are preserved by fibered product". (Base change is another word for fibered products.)

9. Show that closed immersions are monomorphisms.

10. *(quasicompactness is affine-local on the target)* Show that a morphism $f : X \to Y$ is quasicompact if there is cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasicompact. (Hint: affine communication lemma!)

11. Show that the composition of two quasicompact morphisms is quasicompact.

12. *(the notions "locally of finite type" and "finite type" is affine-local on the target)* Show that a morphism $f : X \to Y$ is locally of finite type if there is a cover of Y by open affine sets $\operatorname{Spec} R_i$ such that $f^{-1}(\operatorname{Spec} R_i)$ is locally of finite type over R_i.

13-. Show that a closed immersion is a morphism of finite type.

14-. Show that an open immersion is locally of finite type. Show that an open immersion into a Noetherian scheme is of finite type. More generally, show that a quasicompact open immersion is of finite type.

15-. Show that a composition of two morphisms of finite type is of finite type.

16. Suppose we have a composition of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, where f is quasicompact, and $g \circ f$ is finite type. Show that f is finite type.

17-. Suppose $f : X \to Y$ is finite type, and Y is Noetherian. Show that X is also Noetherian.

18. Suppose X is an affine scheme, and Y is a closed subscheme locally cut out by one equation (e.g. if X is an effective Cartier divisor). Show that $X - Y$ is affine. (This is clear if Y is globally cut out by one equation f; then if $X = \operatorname{Spec} R$ then $Y = \operatorname{Spec} R_f$. However, this is not always true.) Hint: affine locality of the notion of "affine morphism".

19. Here is an explicit consequence of the previous exercise. We showed earlier that on the cone over the smooth quadric surface $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$, the cone over a ruling $w = x = 0$ is not cut out scheme-theoretically by a single equation, by considering Zariski-tangent spaces. We now show that it isn't even cut out set-theoretically by a single equation. For if it were, its complement would be affine. But then the closed subscheme of the complement cut out by $y = z = 0$ would be affine. But this is the scheme $y = z = 0$ (also known as the wx-plane) minus the point $w = x = 0$, which we've seen is non-affine. (For comparison, on the cone $\operatorname{Spec} k[x, y, z]/(xy - z^2)$, the ruling $x = z = 0$ is not cut out scheme-theoretically by a single equation, but it *is* cut out set-theoretically by $x = 0$.) Verify all of this!

20. *(the property of finiteness is affine-local on the target)* Show that a morphism $f : X \to Y$ is finite if there is a cover of Y by open affine sets $\operatorname{Spec} R$ such that $f^{-1}(\operatorname{Spec} R)$ is the spectrum of a finite R-algebra. (Hint: Use that $f_*\mathcal{O}_X$ is finite type.)

21-. Show that closed immersions are finite morphisms.

22. (a) Show that if a morphism is finite then it is quasifinite. (b) Show that the converse is not true. (Hint: $\mathbb{A}^1 - \{0\} \to \mathbb{A}^1$.)

23. Suppose X is a Noetherian scheme. Show that a subset of X is constructable if and only if it is the finite disjoint union of locally closed subsets.

24-. Show that the image of an irreducible scheme is irreducible.

Class 20:

25. Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes, and suppose M is an A-module, so $\tilde{M}$ is a quasicoherent sheaf on $\operatorname{Spec} A$. Show that $f_* \tilde{M} \cong \widetilde{M_B}$. (Hint: There is only one reasonable way to proceed: look at distinguished opens!)

26. Give an example of a morphism of schemes $\pi : X \to Y$ and a quasicoherent sheaf $\mathcal{F}$ on X such that $\pi_* \mathcal{F}$ is not quasicoherent. (Answer: $Y = \mathbb{A}^1$, $X =$ countably many copies of $\mathbb{A}^1$. Then let $f = t$. X_t has a global section $(1/t, 1/t^2, 1/t^3, \dots)$. The key point here is that infinite direct sums do not commute with localization.)

27. Suppose $f : X \to Y$ is a finite morphism of Noetherian schemes. If $\mathcal{F}$ is a coherent sheaf on X, show that $f_* \mathcal{F}$ is a coherent sheaf. (Hint: Show first that $f_* \mathcal{O}_X$ is finite type = locally finitely generated.)

28. Verify that the following is an example showing that pullback is not left-exact: consider the exact sequence of sheaves on $\mathbb{A}^1$, where p is the origin:

$$0 \to \mathcal{O}_{\mathbb{A}^1}(-p) \to \mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_p \to 0.$$

(This is a closed subscheme exact sequence; also an effective Cartier exact sequence. Algebraically, we have $k[t]$-modules $0 \to tk[t] \to k[t] \to k \to 0$.) Restrict to p.

Class 21:

29. The notion of integral morphism is well behaved with localization and quotient of B, and quotient of A (but not localization of A, witness $k[t] \to k[t]$, but $k[t] \to k[t]_{(t)}$). The notion of integral extension is well behaved with respect to localization and quotient of B, but not quotient of A (same example, $k[t] \to k[t]/(t)$).

30+. (a) Show that if B is an integral extension of A, and C is an integral extension of B, then C is an integral extension of A.
(b) Show that if B is a finite extension of A, and C is a finite extension of B, then C is an finite extension of A.

31-. Show that the special case of the going-up theorem where A is a field translates to: if $B \subset A$ is a subring with A integral over B, then B is a field. Prove this. (Hint: all you need to do is show that all nonzero elements in B have inverses in B. Here is the start: If $b \in B$, then $1/b \in A$, and this satisfies some integral equation over B.)

32+. *(sometimes also called the going-up theorem)* Show that if $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n$ is a chain of prime ideals of B, and $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$ is a chain of prime ideals of A such that $\mathfrak{p}_i$ "lies over" $\mathfrak{q}_i$ (and $m < n$), then the second chain can be extended to $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ so that this remains true.

33+. Show that if $f : \operatorname{Spec} A \to \operatorname{Spec} B$ corresponds to an integral *extension of rings*, then $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$.

34. Show that finite morphisms are *closed*, i.e. the image of any closed subset is closed.

35. Show that integral ring extensions induce a surjective map of spectra.

36. Suppose X is a Noetherian scheme. Show that a subset of X is constructable if and only if it is the finite disjoint union of locally closed subsets. (This is admittedly the same as 23.)

37. Show that a dominant morphism of integral schemes $X \to Y$ induces an inclusion of function fields in the other direction.

38. If $\phi : A \to B$ is a ring morphism, show that the corresponding morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is dominant iff ϕ has nilpotent kernel.

39+. Reduce the proof of Chevalley's theorem to the following case: suppose $f : X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ is a dominant morphism, where A and B are domains, and f corresponds to $\phi : B \to B[x_1, \ldots, x_n]/I \cong A$. Show that the image of f contains a dense open subset of $\operatorname{Spec} B$. (See the class notes.)

Class 22:

40. Let $\phi : X \to \mathbb{P}^n_A$ be a morphism of A-schemes, corresponding to an invertible sheaf $\mathcal{L}$ on X and sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ as above. Then ϕ is a closed immersion iff (1) each open set $X_i = X_{s_i}$ is affine, and (2) for each i, the map of rings $A[y_0, \ldots, y_n] \to \Gamma(X_i, \mathcal{O}_{X_i})$ given by $y_j \mapsto s_j/s_i$ is surjective.

41. *(Automorphisms of projective space)* Show that all the automorphisms of projective space $\mathbb{P}^n_k$ correspond to $(n+1) \times (n+1)$ invertible matrices over k, modulo scalars (also known as $PGL_{n+1}(k)$). (Hint: Suppose $f : \mathbb{P}^n_k \to \mathbb{P}^n_k$ is an automorphism. Show that $f^*\mathcal{O}(1) \cong \mathcal{O}(1)$. Show that $f^* : \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ is an isomorphism.)

42. Show that any map from projective space to a smaller projective space is constant. (Fun!)

43. Prove that $\mathbb{A}^n_R \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$. Prove that $\mathbb{P}^n_R \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$.

44. Show that for finite-type schemes over $\mathbb{C}$, the complex-valued points of the fibered product correspond to the fibered product of the complex-valued points. (You will just use the fact that $\mathbb{C}$ is algebraically closed.)

45-. Describe $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$.

46. Consider the morphism of schemes $X = \operatorname{Spec} k[t] \to Y = \operatorname{Spec} k[u]$ corresponding to $k[u] \to k[t]$, $u = t^2$ (where the characteristic of k is not 2). Show that $X \times_Y X$ has 2 irreducible components. Compare what is happening above the generic point of Y to the previous exercise.

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