FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 8

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This set is due Wednesday, December 14, in my mailbox. (I will accept it, and other older sets, until Friday, December 16. That will likely be a hard deadline, because I may not be around to pick up any later sets.) It covers (roughly) classes 17 and 18.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in four solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 17:

1. Show that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime.

2-. Show that if $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} \cap \mathfrak{q}'$.

3-. (reality check) Find all the primary ideals in $\mathbb{Z}$.

4+. Suppose A is a Noetherian ring. Show that every proper ideal $I \neq A$ has a primary decomposition. (Hint: Noetherian induction.)

5. Find a minimal primary decomposition of (x^2, xy).

6+. (a) If $\mathfrak{p}$, $\mathfrak{p}_1$, ..., $\mathfrak{p}_n$ are prime ideals, and $\mathfrak{p} = \cap \mathfrak{p}_i$, show that $\mathfrak{p} = \mathfrak{p}_i$ for some i. (Hint: assume otherwise, choose $f_i \in \mathfrak{p}_i - \mathfrak{p}$, and consider $\prod f_i$.)
(b) If $\mathfrak{p} \supset \cap \mathfrak{p}_i$, then $\mathfrak{p} \supset \mathfrak{p}_i$ for some i.
(c) Suppose $I \subseteq \cup^n \mathfrak{p}_i$. Show that $I \subset \mathfrak{p}_i$ for some i. (Hint: by induction on n.)

7. Show that these associated primes behave well with respect to localization. In other words if A is a Noetherian ring, and S is a multiplicative subset (so, as we've seen, there is an inclusion-preserving correspondence between the primes of $S^{-1}A$ and those primes of A not meeting S), then the associated primes of $S^{-1}A$ are just the associated primes of A not meeting S.

8. Show that the minimal primes of 0 are associated primes. (We have now proved important fact **(1)**.) (Hint: suppose $\mathfrak{p} \supset \cap_{i=1}^n \mathfrak{q}_i$. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} \supset \sqrt{\cap_{i=1}^n \mathfrak{q}_i} = \cap_{i=1}^n \sqrt{\mathfrak{q}_i} = \cap_{i=1}^n \mathfrak{p}_i$, so by Exercise 6(b), $\mathfrak{p} \supset \mathfrak{p}_i$ for some i. If $\mathfrak{p}$ is minimal, then as $\mathfrak{p} \supset \mathfrak{p}_i \supset (0)$, we must have

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$\mathfrak{p} = \mathfrak{p}_i$.) Show that there can be other associated primes that are not minimal. (Hint: see an earlier exercise.)

9. Show that if A is reduced, then the only associated primes are the minimal primes.

10. Verify the inclusions and equalities

$$D = \cup_{x \neq 0}(0 : x) \subseteq \cup_{x \neq 0}\sqrt{(0 : x)} \subseteq D.$$

11. Suppose f and g are two global sections of a Noetherian normal scheme with the same poles and zeros. Show that each is a unit times the other.

Class 18:

12. If $W \subset X$ and $Y \subset Z$ are both open immersions of ringed spaces, show that any morphism of ringed spaces $X \to Y$ induces a morphism of ringed spaces $W \to Z$.

13. Show that morphisms of ringed spaces glue. In other words, suppose X and Y are ringed spaces, $X = \cup_i U_i$ is an open cover of X, and we have morphisms of ringed spaces $f_i : U_i \to Y$ that "agree on the overlaps", i.e. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Show that there is a unique morphism of ringed spaces $f : X \to Y$ such that $f|_{U_i} = f_i$. (Long ago we had an exercise proving this for topological spaces.)

14. (Easy but important) Given a morphism of ringed spaces $f : X \to Y$ with $f(p) = q$, show that there is a map of stalks $(\mathcal{O}_Y)_q \to (\mathcal{O}_X)_p$.

15. If $f^\# : S \to R$ is a morphism of rings, verify that $R_{f^\# s} \cong R \otimes_S S_s$.

16. Show that morphisms of locally ringed spaces glue (Hint: Basically, the proof of the corresponding exercise for ringed spaces works.)

17+ (easy but important) (a) Show that $\operatorname{Spec} R$ is a locally ringed space. (b) The morphism of ringed spaces $f : \operatorname{Spec} R \to \operatorname{Spec} S$ defined by a ring morphism $f^\# S \to R$ is a morphism of locally ringed spaces.

18++ (Important practice!) Make sense of the following sentence: "$\mathbb{A}^{n+1} - \vec{0} \to \mathbb{P}^n$ given by $(x_0, x_1, \dots, x_{n+1}) \mapsto [x_0; x_1; \dots; x_n]$ is a morphism of schemes." Caution: you can't just say where points go; you have to say where functions go. So you'll have to divide these up into affines, and describe the maps, and check that they glue.

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