# FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 7

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**This set is due Wednesday, December 7. It covers (roughly) classes 15 and 16.**

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions, *including # 23*. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

**Class 15:**

You are not allowed to try the next four problems if you already know how to do them!

**1.** $M$ Noetherian implies that any submodule of $M$ is a finitely generated $R$-module. Hence for example if $R$ is a Noetherian ring then finitely generated = Noetherian.

**2.** If $0 \to M' \to M \to M'' \to 0$ is exact, then $M'$ and $M''$ are Noetherian if and only if $M$ is Noetherian. (Hint: Given an ascending chain in $M$, we get two simultaneous ascending chains in $M'$ and $M''$.)

**3.** $A$ Noetherian as an $A$-module implies $A^n$ is a Noetherian $A$-module.

**4.** If $A$ is a Noetherian ring and $M$ is a finitely generated $A$-module, then any submodule of $M$ is finitely generated. (Hint: suppose $M' \hookrightarrow M$ and $A^n \twoheadrightarrow M$. Construct $N$ with

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d] & A^n \arrow[d] \\ M' \arrow[r, hook] & M \end{tikzcd}$$

.)

**5-.** Show $A$ is coherent (as an $A$-module) if and only if the notion of finitely presented agrees with the notion of coherent.

**6.** If $f \in A$, show that if $M$ is a finitely generated (resp. finitely presented, coherent) $A$-module, then $M_f$ is a finitely generated (resp. finitely presented, coherent) $A_f$-module. (Hint: localization is exact.)

**7.** If $(f_1, \dots, f_n) = A$, and $M_{f_i}$ is finitely generated (resp. coherent) $A_{f_i}$-module for all $i$, then $M$ is a finitely generated (resp. coherent) $A$-module.

*Date*: Monday, November 28, 2005. Minor update January 23, 2006.

**8.** *(Exercise on support of a sheaf)* Show that the support of a finite type quasicoherent sheaf on a scheme is a closed subset. (Hint: Reduce to an affine open set. Choose a finite set of generators of the corresponding module.) Show that the support of a quasicoherent sheaf need not be closed. (Hint: If $A = \mathbb{C}[t]$, then $\mathbb{C}[t]/(t-a)$ is an $A$-module supported at $a$. Consider $\oplus_{a \in \mathbb{C}} \mathbb{C}[t]/(t-a)$.)

**9.** *(Exercise on rank)*

(a) If $m_1, \ldots, m_n$ are generators at $P$, they are generators in an open neighborhood of $P$. (Hint: Consider $\operatorname{coker} A^n \xrightarrow{(f_1,\ldots,f_n)} M$ and Exercise 8.)

(b) Show that at any point, $\operatorname{rank}(\mathcal{F} \oplus \mathcal{G}) = \operatorname{rank}(\mathcal{F}) + \operatorname{rank}(\mathcal{G})$ and $\operatorname{rank}(\mathcal{F} \otimes \mathcal{G}) = \operatorname{rank} \mathcal{F} \operatorname{rank} \mathcal{G}$ at any point. (Hint: Show that direct sums and fibered products commute with ring quotients and localizations, i.e. $(M \oplus N) \otimes_R (R/I) \cong M/IM \oplus N/IN$, $(M \otimes_R N) \otimes_R (R/I) \cong (M \otimes_R R/I) \otimes_{R/I} (N \otimes_R R/I) \cong M/IM \otimes_{R/I} N/IM$, etc.) Thanks to Jack Hall for improving this problem.

(c) Show that rank is an upper semicontinuous function on $X$. (Hint: Generators at $P$ are generators nearby.)

**10.** If $X$ is reduced, $\mathcal{F}$ is coherent, and the rank is constant, show that $\mathcal{F}$ is locally free. (Hint: choose a point $p \in X$, and choose generators of the stalk $\mathcal{F}_p$. Let $U$ be an open set where the generators are sections, so we have a map $\phi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$. The cokernel and kernel of $\phi$ are supported on closed sets by Exercise 8. Show that these closed subsets don't include $p$. Make sure you use the reduced hypothesis!) Thus coherent sheaves are locally free on a dense open set. Show that this can be false if $X$ is not reduced. (Hint: $\operatorname{Spec} k[x]/x^2$, $M = k$.)

**11.** *(Geometric Nakayama)* Suppose $X$ is a scheme, and $\mathcal{F}$ is a finite type quasicoherent sheaf. Show that if $\mathcal{F}_x \otimes k(x) = 0$, then there exists $V$ such that $\mathcal{F}|_V = 0$. Better: if I have a set that generates the fiber, it defines the stalk.

**12.** *(Reason for the name "invertible" sheaf)* Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. Then $\mathcal{F}$ and $\mathcal{G}$ are both invertible (Hint: Nakayama.) This is the reason for the adjective "invertible" these sheaves are the invertible elements of the monoid of finite type sheaves. This exercise is a little less important.

**13.** *(A non-quasicoherent sheaf of ideals)* Let $X = \operatorname{Spec} k[x]_{(x)}$, the germ of the affine line at the origin, which has two points, the closed point and the generic point $\eta$. Define $\mathcal{I}(X) = \{0\} \subset \mathcal{O}_X(X) = k[x]_{(x)}$, and $\mathcal{I}(\eta) = k(x) = \mathcal{O}_X(\eta)$. Show that $\mathcal{I}$ is not a quasicoherent sheaf of ideals.

**14.** *(Sections of locally free sheaves cut out closed subschemes)* Suppose $\mathcal{F}$ is a locally free sheaf on a scheme $X$, and $s$ is a section of $\mathcal{F}$. Describe how $s = 0$ "cuts out" a closed subscheme.

**15.** *(Reduction of a scheme)*

(a) $X^{red}$ has the same underlying topological space as X: there is a natural homeomorphism of the underlying topological spaces $X^{red} \cong X$. Picture: taking the reduction may be interpreted as shearing off the fuzz on the space.

(b) Give an example to show that it is *not* true that $\Gamma(X^{red}, \mathcal{O}_{X^{red}}) = \Gamma(X, \mathcal{O}_X)/\sqrt{\Gamma(X, \mathcal{O}_X)}$. (Hint: $\coprod_{n>0} \operatorname{Spec} k[t]/(t^n)$ with global section $(t, t, t, \dots)$.) Motivation for this exercise: this *is* true on each affine open set.

**Class 16:**

**16.** Describe the scheme-theoretic intersection of $(y - x^2)$ and $y$ in $\mathbb{A}^2$. Draw a picture.

**17.** Suppose we have an effective Cartier divisor, a closed subscheme locally cut out by a single equation. As described in class, this gives an invertible sheaf with a canonical section. Show that this section vanishes along our actual effective Cartier divisor.

**18.** Describe the invertible sheaf corresponding to an effective Cartier divisor in terms of transition functions. More precisely, on any affine open set where the effective Cartier divisor is cut out by a single equation, the invertible sheaf is trivial. Determine the transition functions between two such overlapping affine open sets. Verify that there is indeed a canonical section of this invertible sheaf, by describing it.

**19.** Show that $\widetilde{M_*} \otimes \widetilde{N_*} \cong \widetilde{M_* \otimes_{S_*} N_*}$. (Hint: describe the isomorphism of sections over each $D(f)$, and show that this isomorphism behaves well with respect to smaller distinguished opens.)

**20.** *(Closed immersions in projective $S_0$-schemes)* Show that if $I_*$ is a graded ideal of $S_*$, show that we get a closed immersion $\operatorname{Proj} S_*/I_* \hookrightarrow \operatorname{Proj} S_*$.

**21.** Suppose $S_*$ is generated over $S_0$ by $f_1, \dots, f_n$. Suppose $d = \operatorname{lcm}(\deg f_1, \dots, \deg f_n)$. Show that $S_{d*}$ is generated in "new" degree 1 (= "old" degree d). (Hint: I like to show this by induction on the size of the set $\{\deg f_1, \dots, \deg f_n\}$.) This is handy, because we can stick every Proj in some projective space via the construction of previous exercise.

**22.** If $S_*$ is generated in degree 1, show that $\mathcal{O}_{\operatorname{Proj} S_*}(n)$ is an invertible sheaf.

**23. (Mandatory exercise — I am happy to walk you through it, see the notes.)** Calculate $\dim_k \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(n))$.

**24.** Show that $\mathcal{F}(n) \cong \mathcal{F} \otimes \mathcal{O}(n)$.

**25.** Show that $\mathcal{O}(m+n) \cong \mathcal{O}(m) \otimes \mathcal{O}(n)$.

**26.** Show that if $m \neq n$, then $\mathcal{O}_{\mathbb{P}^l_k}(m)$ is not isomorphic to $\mathcal{O}_{\mathbb{P}^l_k}(n)$ if $l > 0$. (Hence we have described a countable number of invertible sheaves (line bundles) that are non-isomorphic. We will see later that these are *all* the line bundles on projective space $\mathbb{P}^n_k$.)

**27.** If quasicoherent sheaves $\mathcal{L}$ and $\mathcal{M}$ are generated by global sections at a point p, then so is $\mathcal{L} \otimes \mathcal{M}$. (This exercise is less important, but is good practice for the concept.)

**28.** An invertible sheaf $\mathcal{L}$ on $X$ is generated by global sections if and only if for any point $x \in X$, there is a section of $\mathcal{L}$ not vanishing at $x$. (Hint: Nakayama.)

**29+. (Important! A theorem of Serre. See the notes for extensive hints.)** Suppose $S_0$ is a Noetherian ring, and $S_*$ is generated in degree 1. Let $\mathcal{F}$ be any finite type sheaf on $\operatorname{Proj} S_*$. Then for some integer $n_0$, for all $n \geq n_0$, $\mathcal{F}(n)$ can be generated by a finite number of global sections.

**30.** Show that $\Gamma_*$ gives a functor from the category of quasicoherent sheaves on $\operatorname{Proj} S_*$ to the category of graded $S_*$-modules. (In other words, show that if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on $\operatorname{Proj} S_*$, describe the natural map $\Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$, and show that such natural maps respect the identity and composition.)

**31.** Show that the canonical map $M_* \to \Gamma_* \widetilde{M_*}$ need not be injective, nor need it be surjective. (Hint: $S_* = k[x]$, $M_* = k[x]/x^2$ or $M_* = \{$ polynomials with no constant terms $\}$.)

**32.** Describe the natural map $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$ as follows. First describe it over $D(f)$. Note that sections of the left side are of the form $m/f^n$ where $m \in \Gamma_{n \deg f} \mathcal{F}$, and $m/f^n = m'/f^{n'}$ if there is some $N$ with $f^N(f^{n'}m - f^n m') = 0$. Show that your map behaves well on overlaps $D(f) \cap D(g) = D(fg)$.

**33+.** Show that the natural map $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$ is an isomorphism, by showing that it is an isomorphism over $D(f)$ for any $f$. Do this by first showing that it is surjective. This will require following some of the steps of the proof of Serre's theorem (a previous exercise on this set). Then show that it is injective. (This is longer, but worth it.)

**34.** *($\Gamma_*$ and $\sim$ are adjoint functors)* Prove part of the statement that $\Gamma_*$ and $\sim$ are adjoint functors, by describing a natural bijection $\operatorname{Hom}(M_*, \Gamma_*(\mathcal{F})) \cong \operatorname{Hom}(\widetilde{M_*}, \mathcal{F})$. For the map from left to right, start with a morphism $M_* \to \Gamma_*(\mathcal{F})$. Apply $\sim$, and postcompose with the isomorphism $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$, to obtain

$$\widetilde{M_*} \to \widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}.$$

Do something similar to get from right to left. Show that "both compositions are the identity in the appropriate category". (Is there a clever way to do that?)

**Coherence:** These twenty problems are only for people who are curious about the notion of coherence for general rings. Others should just skip these. (This is the one exception of my injunction to read all problems.)

**A.** Show that coherent implies finitely presented implies finitely generated.

**B.** Show that $0$ is coherent.

Suppose for problems C–I that

$$0 \to M \to N \to P \to 0 \tag{1}$$

is an exact sequence of $A$-modules.

**Hint $\star$.** Here is a *hint* which applies to several of the problems: try to write

$$\begin{tikzcd}
0 \arrow[r] & A^p \arrow[r] \arrow[d] & A^{p+q} \arrow[r] \arrow[d] & A^q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & N \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

and possibly use the snake lemma.

**C.** Show that N finitely generated implies P finitely generated. (You will only need right-exactness of (1).)

**D.** Show that M, P finitely generated implies N finitely generated. (Possible hint: $\star$.) (You will only need right-exactness of (1).)

**E.** Show that $N, P$ finitely generated need not imply M finitely generated. (Hint: if I is an ideal, we have $0 \to I \to A \to A/I \to 0$.)

**F.** Show that N coherent, M finitely generated implies M coherent. (You will only need left-exactness of (1).)

**G.** Show that N, P coherent implies M coherent. Hint for (i) in the definition of coherence:

$$\begin{tikzcd}
 & A^q \arrow[dr] \arrow[dd] & & & \\
 & & A^p \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[dr] & 0 \\
 & 0 & 0 & & 0
\end{tikzcd}$$

(You will only need left-exactness of (1).)

**H.** Show that M finitely generated and N coherent implies P coherent. (Hint for (ii) in the definition of coherence: $\star$. You will only right-exactness of (1).)

**I.** Show that M, P coherent implies N coherent. (Hint: $\star$.)

At this point, we have shown that if two of (1) are coherent, the third is as well.

**J.** Show that a finite direct sum of coherent modules is coherent.

**K.** Suppose M is finitely generated, N coherent. Then if $\phi : M \to N$ is any map, then show that $\operatorname{Im} \phi$ is coherent.

**L.** Show that the kernel and cokernel of maps of coherent modules are coherent.

At this point, we have verified that coherent $A$-modules form an abelian subcategory of the category of $A$-modules. (Things you have to check: $0$ should be in this set; it should be closed under finite sums; and it should be closed under taking kernels and cokernels.)

**M.** Suppose $M$ and $N$ are coherent submodules of the coherent module $P$. Show that $M+N$ and $M \cap N$ are coherent. (Hint: consider the right map $M \oplus N \to P$.)

**N.** Show that if $A$ is coherent (as an $A$-module) then finitely presented modules are coherent. (Of course, if finitely presented modules are coherent, then $A$ is coherent, as $A$ is finitely presented!) (This is also # 5.)

**O.** If $M$ is finitely presented and $N$ is coherent, show that $\mathrm{Hom}(M, N)$ is coherent. (Hint: $\mathrm{Hom}$ is left-exact in its first entry.)

**P.** If $M$ is finitely presented, and $N$ is coherent, show that $M \otimes N$ is coherent.

**Q.** If $f \in A$, show that if $M$ is a finitely generated (resp. finitely presented, coherent) $A$-module, then $M_f$ is a finitely generated (resp. finitely presented, coherent) $A_f$-module. Hint: localization is exact. (This is also # 6.)

**R.** Suppose $(f_1, \dots, f_n) = A$. Show that if $M_{f_i}$ is finitely generated for all $i$, then $M$ is too. (Hint: Say $M_{f_i}$ is generated by $m_{ij} \in M$ as an $A_{f_i}$-module. Show that the $m_{ij}$ generate $M$. To check surjectivity $\oplus_{i,j} A \to M$, it suffices to check "on $D(f_i)$" for all $i$.) (This is half of # 7.)

**S.** Suppose $(f_1, \dots, f_n) = A$. Show that if $M_{f_i}$ is coherent for all $i$, then $M$ is too. (Hint from Rob Easton: if $\phi : A^2 \to M$, then $(\ker \phi)_{f_i} = \ker(\phi_{f_i})$, which is finitely generated for all $i$. Then apply the previous exercise.) (This is the other half of # 7.)

**T.** Show that the ring $A := k[x_1, x_2, \dots]$ is not coherent over itself. (Hint: consider $A \to A$ with $x_1, x_2, \dots \mapsto 0$.) Thus we have an example of a finitely presented module that is not coherent; a surjection of finitely presented modules whose kernel is not even finitely generated; hence an example showing that finitely presented modules don't form an abelian category.

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