FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 6

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This set is due Wednesday, November 30. It covers (roughly) classes 13 and 14.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 13:

1. Show that $(x, z) \subset k[w, x, y, z]/(wz - xy)$ is a height 1 ideal that is not principal. (Make sure you have a picture of this in your head!)

2. Suppose X is an integral Noetherian scheme, and $f \in \operatorname{Frac}(\Gamma(X, \mathcal{O}_X))^*$ is a non-zero element of its function field. Show that f has a finite number of zeros and poles. (Hint: reduce to $X = \operatorname{Spec} R$. If $f = f_1/f_2$, where $f_i \in R$, prove the result for f_i.)

3. Let R be the subring $k[x^3, x^2, xy, y] \subset k[x, y]$. (The idea behind this example: I'm allowing all monomials in $k[x, y]$ except for x.) Show that it is not integrally closed (easy — consider the "missing x"). Show that it is regular in codimension 1 (hint: show it is dimension 2, and when you throw out the origin you get something nonsingular, by inverting x^2 and y respectively, and considering R_{x^2} and R_y).

4. You have checked that if $k = \mathbb{C}$, then $k[w, x, y, z]/(wx - yz)$ is integrally closed (PS4, problem B5). Show that it is not a unique factorization domain. (The most obvious possibility is to do this "directly", but this might be hard. Another possibility, faster but less intuitive, is to prove the intermediate result that *in a unique factorization domain, any height* 1 *prime is principal*, and considering Exercise 1.)

5. Show that on a Noetherian scheme, you can check nonsingularity by checking at closed points. (Caution: a scheme in general needn't have any closed points!) You are allowed to use the unproved fact from the notes, that any localization of a regular local ring is regular.

6. Show that a nonsingular locally Noetherian scheme is irreducible if and only if it is connected. (I'm not sure if this fact requires Noetherianness.)

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7-. Show that there is a nonsingular hypersurface of degree d. Show that there is a Zariski-open subset of the space of hypersurfaces of degree d. The two previous sentences combine to show that the nonsingular hypersurfaces form a Zariski-open set. Translation: almost all hypersurfaces are smooth.

8-. Suppose $(R, \mathfrak{m}, k)$ is a regular Noetherian local ring of dimension n. Show that $\dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \binom{n+i-1}{i}$.

9. Show that fact 2 in the "good facts to know about regular local rings" implies that $(R, \mathfrak{m})$ is a domain. (Hint: show that if $f, g \neq 0$, then $fg \neq 0$, by considering the leading terms.)

Note that we have proved this fact (referred to in the previous problem) if $(R, \mathfrak{m})$ is a Noetherian local ring containing its residue field k. The next three exercises fill out the proof in the notes. Do them only if you are fairly happy with other things.

10. If S is a Noetherian ring, show that $S[[t]]$ is Noetherian. (Hint: Suppose $I \subset S[[t]]$ is an ideal. Let $I_n \subset S$ be the coefficients of t^n that appear in the elements of I form an ideal. Show that $I_n \subset I_{n+1}$, and that I is determined by $(I_0, I_1, I_2, \dots)$.)

11. Show that $\dim k[[t_1, \dots, t_n]]$ is dimension n. (Hint: find a chain of $n+1$ prime ideals to show that the dimension is at least n. For the other inequality, use Krull.)

12. If R is a Noetherian local ring, show that $\hat{R} := \lim_{\leftarrow} R/\mathfrak{m}^n$ is a Noetherian local ring. (Hint: Suppose $\mathfrak{m}/\mathfrak{m}^2$ is finite-dimensional over k, say generated by $x_1, \dots, x_n$. Describe a surjective map $k[[t_1, \dots, t_n]] \to \hat{R}$.)

13. Show that a section of a sheaf on the distinguished affine base is determined by the section's germs.

14+. Recall Theorem 4.2(a) in the class 13 notes, which states that a sheaf on the distinguished affine base $\mathcal{F}^b$ determines a unique sheaf $\mathcal{F}$, which when restricted to the affine base is $\mathcal{F}^b$. We defined

$$\mathcal{F}(U) := \{(f_x \in \mathcal{F}^b_x)_{x \in U} : \forall x \in U, \exists U_X \text{ with } x \subset U_x \subset U, F^x \in \mathcal{F}^b(U_x) : F^x_y = f_y \forall y \in U_x\}$$

where each U_x is in our base. In class I claimed that if U is in our base, that $\mathcal{F}(U) = \mathcal{F}^b(U)$. We clearly have a map $\mathcal{F}^b(U) \to \mathcal{F}(U)$. Prove that it is an isomorphism.

15+. Show that a sheaf of $\mathcal{O}_X$-modules on "the distinguished affine base" yields an $\mathcal{O}_X$-module.

Class 14:

16+. (a first example of the total complex of a double complex) Suppose $0 \to A \to B \to C$ is exact. Define the total complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle -\mathrm{id}} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \end{array}$$

as

$$0 \to A \to A \oplus B \to B \oplus C$$

in the "obvious" way. Show that the total complex is also exact.

17. (a) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the origin $[(t)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this is not a quasicoherent sheaf. (More generally, if X is an integral scheme, and $p \in X$ that is not the generic point, we could take the skyscraper sheaf at p with group the function field of X. Except in a silly circumstances, this sheaf won't be quasicoherent.)
(b) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the generic point $[(0)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this *is* a quasicoherent sheaf. Describe the restriction maps in the distinguished topology of X.

18+. (Important Exercise for later) Suppose X is a Noetherian scheme. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X, and let $f \in \Gamma(X, \mathcal{O}_X)$ be a function on X. Let $R = \Gamma(X, \mathcal{O}_X)$ for convenience. Show that the restriction map $\operatorname{res}_{X_f \subset X} : \Gamma(X, \mathcal{F}_X) \to \Gamma(X_f, \mathcal{F}_X)$ (here X_f is the open subset of X where f doesn't vanish) is precisely localization. In other words show that there is an isomorphism $\Gamma(X, \mathcal{F})_f \to \Gamma(X_f, \mathcal{F})$ making the following diagram commute.

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \xrightarrow{\operatorname{res}_{X_f \subset X}} & \Gamma(X_f, \mathcal{F}) \\ & {\scriptstyle \otimes_R R_f} \searrow \quad \nearrow {\scriptstyle \sim} & \\ & \Gamma(X, \mathcal{F})_f & \end{array}$$

All that you should need in your argument is that X admits a cover by a finite number of open sets, and that their pairwise intersections are each quasicompact. We will later rephrase this as saying that X is quasicompact and quasiseparated. (Hint: cover by affine open sets. Use the sheaf property. A nice way to formalize this is the following. Apply the exact functor $\otimes_R R_f$ to the exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \oplus_i \Gamma(U_i, \mathcal{F}) \to \oplus \Gamma(U_{ijk}, \mathcal{F})$$

where the U_i form a finite cover of X and U_{ijk} form an affine cover of $U_i \cap U_j$.)

19-. Give a counterexample to show that the above statement need not hold if X is not quasicompact. (Possible hint: take an infinite disjoint union of affine schemes.)

20. (This is for arithmetically-minded people only — I won't define my terms.) Prove that a fractional ideal on a ring of integers in a number field yields an invertible sheaf. Show that any two that differ by a principal ideal yield the same invertible sheaf. (Thus we have described a map from the class group of the number field to the Picard group of its ring of integers. We will later see that this is an isomorphism.)

21+. Show that you can check exactness of a sequence of quasicoherent sheaves on an affine cover. (In particular, taking sections over an affine open $\operatorname{Spec} R$ is an exact functor from the category of quasicoherent sheaves on X to the category of R-modules. Recall that taking sections is only left-exact in general. Similarly, you can check surjectivity on an affine cover unlike sheaves in general.)

22+. If $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is given by the following information: If $\operatorname{Spec} R$ is an affine open, and $\Gamma(\operatorname{Spec} R, \mathcal{F}) = M$ and $\Gamma(\operatorname{Spec} R, \mathcal{G}) = N$, then $\Gamma(\operatorname{Spec} R, \mathcal{F} \otimes \mathcal{G}) = M \otimes N$, and the restriction map $\Gamma(\operatorname{Spec} R, \mathcal{F} \otimes \mathcal{G}) \to \Gamma(\operatorname{Spec} R_f, \mathcal{F} \otimes \mathcal{G})$ is precisely the localization map $M \otimes_R N \to (M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. (We are using the algebraic fact that that $(M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. You can prove this by universal property if you want, or by using the explicit construction.)

23. If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is locally free. (Possible hint for this, and later exercises: check on sufficiently small affine open sets.)

24. Prove the following.
(a) Tensoring by a quasicoherent sheaf is right-exact. More precisely, if $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is an exact sequence of quasicoherent sheaves, then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F} \to 0$ is exact.
(b) Tensoring by a locally free sheaf is exact. More precisely, if $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$ is an exact sequence of quasicoherent sheaves, then then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F}$.
(c) The stalk of the tensor product of quasicoherent sheaves at a point is the tensor product of the stalks.
(d) Invertible sheaves on a scheme X (up to isomorphism) form a group. This is called the Picard group of X, and is denoted $\operatorname{Pic} X$. For arithmetic people: this group, for the Spec of the ring of integers R in a number field, is the class group of R.

25. Show that sheaf Hom, $\underline{\operatorname{Hom}}$, is quasicoherent, and is what you think it might be. (Describe it on affine opens, and show that it behaves well with respect to localization with respect to f. To show that $\operatorname{Hom}_A(M, N)_f \cong \operatorname{Hom}_{A_f}(M_f, N_f)$, take a "partial resolution" $A^q \to A^p \to M \to 0$, and apply $\operatorname{Hom}(\cdot, N)$ and localize.) ($\underline{\operatorname{Hom}}$ was defined earlier, and was the subject of a homework problem.) Show that $\underline{\operatorname{Hom}}$ is a left-exact functor in both variables.

26+. Show that if $\mathcal{F}$ is locally free then $\mathcal{F}^\vee$ is locally free, and that there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$. (Caution: your argument showing that if there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$ better not also show that there is a canonical isomorphism $\mathcal{F}^\vee \cong \mathcal{F}$! We'll see an example soon of a locally free $\mathcal{F}$ that is not isomorphic to its dual. The example will be the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$.)

27. The direct sum of quasicoherent sheaves is what you think it is.

For the next exercises, recall the following. If M is an A-module, then the *tensor algebra* $T^*(M)$ is a non-commutative algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as follows. $T^0(M) = A$, $T^n(M) = M \otimes_A \cdots \otimes_A M$ (where n terms appear in the product), and multiplication is what you expect. The *symmetric algebra* $\operatorname{Sym}^* M$ is a symmetric algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as the quotient of $T^*(M)$ by the (two-sided) ideal generated by all elements of the form $x \otimes y - y \otimes x$ for all $x, y \in M$. Thus $\operatorname{Sym}^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - m_1' \otimes \cdots \otimes m_n'$ where $(m_1', \ldots, m_n')$ is a rearrangement of $(m_1, \ldots, m_n)$. The *exterior algebra* $\wedge^* M$ is defined to be the quotient of $T^* M$ by the (two-sided) ideal generated by all elements of the form $x \otimes y + y \otimes x$ for all $x, y \in M$. Thus $\wedge^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - (-1)^{\operatorname{sgn}} m_1' \otimes \cdots \otimes m_n'$ where $(m_1', \ldots, m_n')$ is a rearrangement of $(m_1, \ldots, m_n)$, and the sgn is even if the rearrangement is an even permutation, and odd if the rearrangement is an odd permutation. (It is a "skew-commutative" A-algebra.) It is most correct to write $T^*_A(M)$, $\operatorname{Sym}^*_A(M)$, and $\wedge^*_A(M)$, but the "base ring" is usually omitted for convenience.

28. If $\mathcal{F}$ is a quasicoherent sheaf, then define the quasicoherent sheaves $T^n \mathcal{F}$, $\operatorname{Sym}^n \mathcal{F}$, and $\wedge^n \mathcal{F}$. If $\mathcal{F}$ is locally free of rank m, show that $T^n \mathcal{F}$, $\operatorname{Sym}^n \mathcal{F}$, and $\wedge^n \mathcal{F}$ are locally free, and find their ranks.

29+. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves, then for any r, there is a filtration of $\operatorname{Sym}^r \mathcal{F}$:

$$\operatorname{Sym}^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong (\operatorname{Sym}^p \mathcal{F}') \otimes (\operatorname{Sym}^{r-p} \mathcal{F}'')$$

for each p.

30. Suppose $\mathcal{F}$ is locally free of rank n. Then $\wedge^n \mathcal{F}$ is called the *determinant (line) bundle*. Show that $\wedge^r \mathcal{F} \times \wedge^{n-r} \mathcal{F} \to \wedge^n \mathcal{F}$ is a perfect pairing for all r.

31+. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves, then for any r, there is a filtration of $\wedge^r \mathcal{F}$:

$$\wedge^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong (\wedge^p \mathcal{F}') \otimes (\wedge^{r-p} \mathcal{F}'')$$

for each p. In particular, $\det \mathcal{F} = (\det \mathcal{F}') \otimes (\det \mathcal{F}'')$.

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