FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 5

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This set is due Monday, November 14. It covers (roughly) classes 10, 11, and 12.

As you might have noticed, last week there were a **lot** of interesting problems worth trying — too many to do! (This is just because we've gone far enough that we can really explore interesting questions.) So please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. **Problems marked with "-" count for half a solution.** Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems.

Class 8:

1. (a) Use dimension theory to prove a microscopically stronger version of the weak Nullstellensatz: Suppose $R = k[x_1, \dots, x_n]/I$, where k is an algebraically closed field and I is some ideal. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$.
(b) Suppose $R = k[x_1, \dots, x_n]/I$ where k is not necessarily algebraically closed. Show that every maximal ideal of R has a residue field that is a finite extension of k. [Hint for both: the maximal ideals correspond to dimension 0 points, which correspond to transcendence degree 0 extensions of k, i.e. finite extensions of k. If k is algebraically closed, the maximal ideals correspond to surjections $f : k[x_1, \dots, x_n] \to k$. Fix one such surjection. Let $a_i = f(x_i)$, and show that the corresponding maximal ideal is $(x_1 - a_1, \dots, x_n - a_n)$.]

Class 10:

2+. Suppose R is a ring, and $(f_1, \dots, f_n) = R$. Suppose A is a ring, and R is an A-algebra. Show that if each R_{f_i} is a finitely-generated A-algebra, then so is R.

3. Show that an irreducible homogeneous polynomial in $n+1$ variables (over a field k) describes an integral scheme of dimension $n-1$. We think of this as a "hypersurface in $\mathbb{P}^n_k$".

4. Show that $wx = yz, x^2 = wy, y^2 = xz$ describes an irreducible curve in $\mathbb{P}^3_k$ (the twisted cubic!).

5. Suppose S_* is a graded ring (with grading $\mathbb{Z}^{\geq 0}$). It is automatically a module over S_0. Now $S_+ := \oplus_{i>0} S_i$ is an ideal, which we will call the *irrelevant ideal*; suppose that it is a finitely generated ideal. Show that S_* is a finitely-generated S_0-algebra.

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6+. Recall the definition of the distinguished open subset $D(f)$ on $\operatorname{Proj} S_*$, where f is homogeneous of positive degree. Show that

$$(D(f), \mathcal{O}_{\operatorname{Proj} S_*}) \cong \operatorname{Spec}(S_f)_0$$

defines a sheaf on $\operatorname{Proj} S_*$. (We used this to define the structure sheaf $\mathcal{O}_{\operatorname{Proj} S_*}$ on $\operatorname{Proj} S_*$.)

7-. Show that $\operatorname{Proj} k[x_0, \ldots, x_n]$ is isomorphic to our earlier definition of $\mathbb{P}^n$.

8-. Show that $Y = \mathbb{P}^2 - (x^2 + y^2 + z^2 = 0)$ is affine, and find its corresponding ring (= find its ring of global sections).

Class 11:

9-. Show that $\mathbb{P}^0_A = \operatorname{Proj} A[T] \cong A$. Thus "$\operatorname{Spec} A$ is a projective A-scheme".

10. Show that all projective A-schemes are quasicompact. (Translation: show that any projective A-scheme is covered by a finite number of affine open sets.) Show that $\operatorname{Proj} S_*$ is finite type over $A = S_0$. If S_0 is a Noetherian ring, show that $\operatorname{Proj} S_*$ is a Noetherian scheme, and hence that $\operatorname{Proj} S_*$ has a finite number of irreducible components. Show that any quasiprojective scheme is locally of finite type over A. If A is Noetherian, show that any quasiprojective A-scheme is quasicompact, and hence of finite type over A.

11. Give an example of a quasiprojective A-scheme that is not quasicompact (necessarily for some non-Noetherian A).

12-. Show that $\mathbb{P}^n_k$ is normal. More generally, show that $\mathbb{P}^n_R$ is normal if R is a Unique Factorization Domain.

13+. Show that the projective cone of $\operatorname{Proj} S_*$ has an open subscheme $D(T)$ that is the affine cone, and that its complement $V(T)$ can be identified with $\operatorname{Proj} S_*$ (as a topological space). (More precisely, setting $T = 0$ "cuts out" a scheme isomorphic to $\operatorname{Proj} S_*$ — see if you can make that precise.)

14. If S_* is a finitely generated domain over k, and $\operatorname{Proj} S_*$ is non-empty show that $\dim \operatorname{Spec} S_* = \dim \operatorname{Proj} S_* + 1$.

15. Show that the irreducible subsets of dimension $n-1$ of $\mathbb{P}^n_k$ correspond to homogeneous irreducible polynomials modulo multiplication by non-zero scalars.

16+.

(a) Suppose I is any homogeneous ideal, and f is a homogeneous element. Suppose f vanishes on $V(I)$. Show that $f^n \in I$ for some n. (Hint: Mimic the proof in the affine case.)
(b) If $Z \subset \operatorname{Proj} S_*$, define $I(\cdot)$. Show that it is a homogeneous ideal. For any two subsets, show that $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.
(c) For any homogeneous ideal I with $V(I) \neq \emptyset$, show that $I(V(I)) = \sqrt{I}$.
(d) For any subset $Z \subset \operatorname{Proj} S_*$, show that $V(I(Z)) = \overline{Z}$.

17. Show that the following are equivalent. (a) $V(I) = \emptyset$ (b) for any f_i (i in some index set) generating I, $\cup D(f_i) = \operatorname{Proj} S_*$ (c) $\sqrt{I} \supset S_+$.

18+. Show that $\operatorname{Proj} S_{n\cdot}$ is isomorphic to $\operatorname{Proj} S_*$.

For problems 19-21, suppose $S_* = k[x, y]$ (with the usual grading).

19. Show that $S_{2\cdot} \cong k[u, v, w]/(uw - v^2)$. (Thus the 2-uple Veronese embedding of $\mathbb{P}^1$ is as a conic in $\mathbb{P}^2$.)

20. Show that $\operatorname{Proj} S_{3\cdot}$ is the *twisted cubic* "in" $\mathbb{P}^3$. (The equations of the twisted cubic turn up in problems 4 and 39.)

21+. Show that $\operatorname{Proj} S_{d\cdot}$ is given by the equations that

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix}$$

is rank 1 (i.e. that all the 2×2 minors vanish). This is called the *degree* d *rational normal curve* "in" $\mathbb{P}^d$.

22. Find the equations cutting out the *Veronese surface* $\operatorname{Proj} S_{2\cdot}$ where $S_* = k[x_0, x_1, x_2]$, which sits naturally in $\mathbb{P}^5$.

23. Show that $\mathbb{P}(m, n)$ is isomorphic to $\mathbb{P}^1$. Show that $\mathbb{P}(1, 1, 2) \cong \operatorname{Proj} k[u, v, w, z]/(uw - v^2)$. Hint: do this by looking at the even-graded parts of $k[x_0, x_1, x_2]$. (Picture: this is a projective cone over a conic curve.)

24+. (This is a handy exercise for later.) (a) *(Hypersurfaces meet everything of dimension at least* 1 *in projective space — unlike in affine space.)* Suppose X is a closed subset of $\mathbb{P}^n_k$ of dimension at least 1, and H a nonempty hypersurface in $\mathbb{P}^n_k$. Show that H meets X. (Hint: consider the affine cone, and note that the cone over H contains the origin. Use Krull's Principal Ideal Theorem.)
(b) (Definition: Subsets in $\mathbb{P}^n$ cut out by linear equations are called *linear subspaces*. Dimension 1, 2 linear subspaces are called *lines* and *planes* respectively.) Suppose $X \hookrightarrow \mathbb{P}^n_k$ is a closed subset of dimension r. Show that any codimension r linear space meets X. (Hint: Refine your argument in (a).)
(c) Show that there is a codimension $r + 1$ complete intersection (codimension $r + 1$ set that is the intersection of $r + 1$ hypersurfaces) missing X. (The key step: show that there is a hypersurface that doesn't contain every generic point of X.) If k is infinite, show that there is a codimension $r + 1$ linear subspace missing X. (The key step: show that there is a hyperplane not containing any generic point of a component of X.)

25. Describe all the lines on the quadric surface $wx - yz = 0$ in $\mathbb{P}^3_k$. (Hint: they come in two "families", called the *rulings* of the quadric surface.)

26. (This is intended for people who already know what derivations are.) In differential geometry, the tangent space at a point is sometimes defined as the vector space of derivations at that point. A derivation is a function that takes in functions near the point that vanish at the point, and gives elements of the field k, and satisfies the Leibniz rule

$(fg)' = f'g + g'f$. Show that this agrees with our definition of tangent space. (One direction was shown in class 11.)

27+. *(Nakayama's lemma version 3)* Suppose R is a ring, and I is an ideal of R contained in all maximal ideals. Suppose M is a *finitely generated* R-module, and $N \subset M$ is a submodule. If $N/IN \hookrightarrow M/IM$ an isomorphism, then $M = N$.

28+. *(Nakayama's lemma version 4)* Suppose $(R, \mathfrak{m})$ is a local ring. Suppose M is a finitely-generated R-module, and $f_1, \dots, f_n \in M$, with (the images of) $f_1, \dots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \dots, f_n$ generate M. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.)

Class 12:

29-. Show that if A is a Noetherian local ring, then A has finite dimension. (*Warning:* Noetherian rings in general could have infinite dimension.)

30+. *(the Jacobian criterion for checking nonsingularity)* Suppose k is an algebraically closed field, and X is a finite type k-scheme. Then locally it is of the form $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Show that the Zariski tangent space at the closed point p (with residue field k, by the Nullstellensatz) is given by the cokernel of the Jacobian map $k^r \to k^n$ given by the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix}. \tag{1}$$

(This is just making precise our example of a curve in $\mathbb{A}^3$ cut out by a couple of equations, where we picked off the linear terms .) Possible hint: "translate p to the origin," and consider linear terms.

31. Show that the singular *closed* points of the hypersurface $f(x_1, \dots, x_n) = 0$ in $\mathbb{A}^n_k$ are given by the equations $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$.

32. Show that $\mathbb{A}^1$ and $\mathbb{A}^2$ are nonsingular. (Make sure to check nonsingularity at the non-closed points! Fortunately you know what all the points of $\mathbb{A}^2$ are; this is trickier for $\mathbb{A}^3$.) You are not allowed to use the fact that regular local rings remain regular under localization.

33. Show that $\operatorname{Spec} \mathbb{Z}$ is a nonsingular curve.

34. Note that $\mathbb{Z}[i]$ is dimension 1, as $\mathbb{Z}[x]$ has dimension 2 (problem set exercise), and is a domain, and $x^2 + 1$ is not 0, so $\mathbb{Z}[x]/(x^2 + 1)$ has dimension 1 by Krull. Show that $\operatorname{Spec} \mathbb{Z}[i]$ is a nonsingular curve. (This exercise is intended for people who know about the primes in the Gaussian integers $\mathbb{Z}[i]$.)

35. Show that there is one singular point of $\operatorname{Spec} \mathbb{Z}[2i]$, and describe it.

36. *(the Euler test for projective hypersurfaces)* There is an analogous Jacobian criterion for hypersurfaces $f = 0$ in $\mathbb{P}^n_k$. Show that the singular *closed* points correspond to the locus $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. If the degree of the hypersurface is not divisible by the characteristic of any of the residue fields (e.g. if we are working over a field of characteristic 0), show that it suffices to check $\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. (Hint: show that f lies in the ideal $(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$). (Fact: this will give the singular points in general. I don't want to prove this, and I won't use it.)

37-. Suppose k is algebraically closed. Show that $y^2 z = x^3 - xz^2$ in $\mathbb{P}^2_k$ is an irreducible nonsingular curve. (This is for practice.) Warning: I didn't say $\operatorname{char} k = 0$.

38-. Find all the singular closed points of the following plane curves. Here we work over a field of characteristic 0 for convenience.

(a) $y^2 = x^2 + x^3$. This is called a *node*.
(b) $y^2 = x^3$. This is called a *cusp*.
(c) $y^2 = x^4$. This is called a *tacnode*.

39. Show that the twisted cubic $\operatorname{Proj} k[w, x, y, z]/(wz - xy, wy - x^2, xz - y^2)$ is nonsingular. (You can do this by using the fact that it is isomorphic to $\mathbb{P}^1$. I'd prefer you to do this with the explicit equations, for the sake of practice.)

40-. Show that the only dimension 0 Noetherian regular local rings are fields. (Hint: Nakayama.)

41-. Consider the following two examples:
(i) *(the 5-adic valuation)* $K = \mathbb{Q}$, $\nu(r)$ is the "power of 5 appearing in r", e.g. $\nu(35/2) = 1$, $\nu(27/125) = -3$.
(ii) $K = k(x)$, $\nu(f)$ is the "power of x appearing in f.
Describe the valuation rings in those two examples.

42. Show that $0 \cup \{x \in K^* : \nu(x) \geq 1\}$ is the unique maximal ideal of the valuation ring. (Hint: show that everything in the complement is invertible.) Thus the valuation ring is a local ring.

43+. Show that every discrete valuation ring is a Noetherian regular local ring of dimension 1. (This was part of our long theorem showing that many things were equivalent.)

44-. Suppose R is a Noetherian local domain of dimension 1. Show that R is a principal ideal domain if and only if it is a discrete valuation ring.

45-. Show that there is only one discrete valuation on a discrete valuation ring.

46. Suppose X is a regular integral Noetherian scheme, and $f \in \operatorname{Frac}(\Gamma(X, \mathcal{O}_X))^*$ is a nonzero element of its function field. Show that f has a finite number of zeros and poles.

47+. Suppose A is a subring of a ring B, and $x \in B$. Suppose there is a faithful $A[x]$-module M that is finitely generated as an A-module. Show that x is integral over A. (Hint: look carefully at the proof of Nakayama's Lemma version 1 in the Class 11 notes, and change a few words.)

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