FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 3

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This set is due Monday, October 31. It covers classes 5, 6, and 7. Read all of these problems, and hand in six solutions. Try to solve problems on a range of topics. If you are pressed for time, try more straightforward problems. If you are ambitious, push the envelope a bit. You are encouraged to talk to each other about the problems. (Write up your solutions individually.) You are also encouraged to talk to me about them. Ideally, you should find out who did problems that you didn't do. Make sure you read all the problems, because we will be be making use of many of these results.

Facts we'll use (short proofs)

Three of these count for one problem.

A1. Show that if (S) is the ideal generated by S, then $V(S) = V((S))$. Thus when looking at vanishing sets, it suffices to consider vanishing sets of ideals.

A2. (a) Show that $\emptyset$ and $\operatorname{Spec} R$ are both open.
(b) (The intersection of two open sets is open.) Check that $V(I_1 I_2) = V(I_1) \cup V(I_2)$.
(c) (The union of any collection of open sets is open.) If I_i is a collection of ideals (as i runs over some index set), check that $V(\sum_i I_i) = \cap_i V(I_i)$.

A3. If $I \subset R$ is an ideal, show that $V(\sqrt{I}) = V(I)$.

A4. Show that if R is an integral domain, then $\operatorname{Spec} R$ is an irreducible topological space. (Hint: look at the point $[(0)]$.)

A5. Show that the closed points of $\operatorname{Spec} R$ correspond to the maximal ideals.

A6. If $X = \operatorname{Spec} R$, show that $[\mathfrak{p}]$ is a specialization of $[\mathfrak{q}]$ if and only if $\mathfrak{q} \subset \mathfrak{p}$.

A7. If X is a finite union of quasicompact spaces, show that X is quasicompact.

A8. Suppose $f_i \in R$ for $i \in I$. Show that $\cup_{i \in I} D(f_i) = \operatorname{Spec} R$ if and only if $(f_i) = R$.

A9. Show that $D(f) \cap D(g) = D(fg)$. Hence the distinguished base is a *nice* base.

A10. Show that if $D(f) \subset D(g)$, then $f^n \in (g)$ for some n.

A11. Show that $f \in \mathfrak{N}$ if and only if $D(f) = \emptyset$.

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A12. Suppose $f \in R$. Show that under the identification of $D(f)$ in $\operatorname{Spec} R$ with $\operatorname{Spec} R_f$, there is a natural isomorphism of sheaves $(D(f), \mathcal{O}_{\operatorname{Spec} R}|_{D(f)}) \cong (\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f})$.

A13. Show that the disjoint union of a *finite* number of affine schemes is also an affine scheme. (Hint: say what the ring is.)

A14. An infinite disjoint union of (non-empty) affine schemes is not an affine scheme. (One-word hint: quasicompactness.)

A15. If X is a scheme, and U is any open subset, then prove that $(U, \mathcal{O}_X|_U)$ is also a scheme.

A16. Show that if X is a scheme, then the affine open sets form a base for the Zariski topology. (Warning: they don't form a nice base, as we'll see in a different exercise on this problem set.) However, in "most nice situations" this will be true, as we will later see, when we define the analogue of "Hausdorffness", called separatedness.)

Facts we'll use

B1. Show that $\operatorname{Spec} R$ is quasicompact.

B2. Suppose that $I, S \subset R$ are an ideal and multiplicative subset respectively. Show that the Zariski topology on $\operatorname{Spec} R/I$ (resp. $\operatorname{Spec} S^{-1}R$) is the subspace topology induced by inclusion in $\operatorname{Spec} R$. (Hint: compare closed subsets.)

B3. (a) Show that $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set S. (b) Show that if $I \subset R$ is an ideal, then $I(V(I)) = \sqrt{I}$.

B4. (Important!) Show that V and I give a bijection between *irreducible closed subsets* of $\operatorname{Spec} R$ and *prime* ideals of R. From this conclude that in $\operatorname{Spec} R$ there is a bijection between points of $\operatorname{Spec} R$ and irreducible closed subsets of $\operatorname{Spec} R$ (where a point determines an irreducible closed subset by taking the closure). Hence each irreducible closed subset has precisely one generic point.

B5. (Important!) Show that the distinguished opens form a base for the Zariski topology.

B6. (a) Recall that sections of the structure sheaf on the base were defined by $\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f$. Verify that this is well-defined, i.e. if $D(f) = D(f')$ then $R_f \cong R_{f'}$.
(b) Recall that restriction maps on the base were defined as follows. If $D(f) \subset D(g)$, then we have shown that $f^n \in (g)$, i.e. we can write $f^n = ag$, so there is a natural map $R_g \to R_f$ given by $r/g^m \mapsto (ra^m)/(f^{mn})$, and we define

$$\operatorname{res}_{D(g),D(f)} : \mathcal{O}_{\operatorname{Spec} R(D(g))} \to \mathcal{O}_{\operatorname{Spec} R(D(f))}$$

to be this map. Show that $\operatorname{res}_{D(g),D(f)}$ is well-defined, i.e. that it is independent of the choice of a and n, and if $D(f) = D(f')$ and $D(g) = D(g')$, then

$$\begin{array}{ccc} R_g & \xrightarrow{\operatorname{res}_{D(g),D(f)}} & R_f \\ \downarrow \sim & & \downarrow \sim \\ R_{g'} & \xrightarrow{\operatorname{res}_{D(g),D(f)}} & R_{f'} \end{array}$$

commutes.

B7. Show that the structure sheaf satisfies "identity on the distinguished base". Show that it satisfies "gluability on the distinguished base". (We used this to show that the structure sheaf is actually a sheaf.)

B8. Suppose M is an R-module. Show that the following construction describes a sheaf $\tilde{M}$ on the distinguished base. To $D(f)$ we associate $M_f = M \otimes_R R_f$; the restriction map is the "obvious" one.

B9. Show that the stalk of $\mathcal{O}_{\operatorname{Spec} R}$ at the point $[\mathfrak{p}]$ is the ring $R_{\mathfrak{p}}$. (Hint: use distinguished open sets in the direct limit you use to define the stalk. In the course of doing this, you'll discover a useful principle. In the concrete definition of stalk, the elements were sections of the sheaf over *some* open set containing our point, and two sections over different open sets were considered the same if they agreed on some smaller open set. In fact, you can just consider elements of your base when doing this. I think this is called a cofinal system in the directed set, but I might be mistaken.) This is yet another reason to like the notion of a sheaf on a base.

B10. (Important!) Figure out how to define projective n-space $\mathbb{P}^n_k$. Glue together $n+1$ opens each isomorphic to $\mathbb{A}^n_k$. Show that the only global sections of the structure sheaf are the constants, and hence that $\mathbb{P}^n_k$ is not affine if $n > 0$. (Hint: you might fear that you will need some delicate interplay among all of your affine opens, but you will only need two of your opens to see this. There is even some geometric intuition behind this: the complement of the union of two opens has codimension 2. But "Hartogs' Theorem" says that any function defined on this union extends to be a function on all of projective space. Because we're expecting to see only constants as functions on all of projective space, we should already see this for this union of our two affine open sets.)

Practice with the concepts

C1. Verify that $[(y - x^2)] \in \mathbb{A}^2_k$ is a generic point for $V(y - x^2)$.

C2. Suppose $X \subset \mathbb{A}^3_k$ is the union of the three axes. Give generators for the ideal $I(X)$.

C3. Describe a natural isomorphism $(k[x,y]/(xy))_x \cong k[x]_x$.

C4. Suppose we have a polynomial $f(x) \in k[x]$. Instead, we work in $k[x,\epsilon]/\epsilon^2$. What then is $f(x+\epsilon)$? (Do a couple of examples, and you will see the pattern. For example, if $f(x) = 3x^3 + 2x$, we get $f(x+\epsilon) = (3x^3 + 2x) + \epsilon(9x^2 + 2)$. Prove the pattern!) Useful tip: the dual numbers are a good source of (counter)examples, being the "smallest ring with nilpotents". They will also end up being important in defining differential information.

C5. Show that the affine base of the Zariski topology isn't necessarily a nice base. (Hint: look at the affine plane with the doubled origin.)

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