FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 1

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This set is due Monday, October 17. It covers classes 1 and 2. Hand in five of these problems. If you are pressed for time, try more A problems. If you are ambitious, try more B problems.

I intend there to be weekly problem sets, to be given out each Monday and handed in the following Monday (although this set is an exception). If you are taking this course for a grade, you'll have to hand in all but one of the sets. These problems are not intended to be (too) onerous, but they are intended to help you get practical experience with ideas that may be new to you. Even if you are not taking the course for a grade, I strongly encourage you to try these problems, and if you are handing in problems, I encourage you to try more than the minimum number. Choose problems that stretch your knowledge, and not problems that you already know how to do. Feedback on the problems would be appreciated.

You are encouraged to talk to each other about the problems. (Write up your solutions individually.) You are also encouraged to talk to me about them. Ideally, you should find out who did problems that you didn't do.

I will be away Wednesday, October 5 until Thursday, October 13. The next class after Monday, October 3 will be Friday, October 14. *The week after we will meet Monday, Wednesday, and Friday (Oct. 17, 19, 21). Then we will be only one class behind.*

A1. A category in which each morphism is an isomorphism is called a *groupoid*. A perverse definition of a group is: a groupoid with one element. Make sense of this. (The notion of "groupoid" isn't important for this course. The point of this exercise is to give you some practice with categories, by relating them to an object you know well.)

A2 (if you haven't seen tensor products before). Calculate $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/12$. (The point of this exercise is to give you a very little hands-on practice with tensor products.)

A3. Interpret fibered product in the category of sets: If we are given maps from sets X and Y to the set Z, interpret $X \times_Z Y$. (This will help you build intuition about this concept.)

A4. A morphism $f : X \to Y$ is said to be a a **monomorphism** if any two morphisms $g_1, g_2 : Z \to X$ such that $f \circ g_1 = f \circ g_2$ must satisfy $g_1 = g_2$. This is the generalization of an injection of sets. Suppose $X \to Y$ is a monomorphism, and $W, Z \to X$ are two morphisms. Show that $W \times_X Z$ and $W \times_Y Z$ are canonically isomorphic. (We will use this later when talking about fibered products.)

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A5. Given $X \to Y \to Z$, show that there is a natural morphism $X \times_Y X \to X \times_Z X$, assuming these fibered products exist. (This is trivial once you figure out what it is saying. The point of this exercise is to see why it is trivial.)

A6. Define coproduct in a category by reversing all the arrows in the definition of product. Show that coproduct for sets is disjoint union.

A7. If Z is the final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, then "$X \times_Z Y = X \times Y$" ("the" fibered product over Z is canonically isomorphic to "the" product). (This is an exercise about unwinding the definition.)

A8 ("A presheaf is the same as a contravariant functor"). Given any topological space X, we can get a category, which I will call the "category of open sets". The objects are the open sets. The morphisms are the inclusions $U \hookrightarrow V$. (What is the initial object? What is the final object?) Verify that the data of a presheaf is precisely the data of a contravariant functor from the category of open sets of X to the category of sets, plus the final object axiom, that there is one section over $\emptyset$. (This exercise is intended for people wanting practice with categories.)

A9. (a) Let X be a topological space, and S a set with more than one element, and define $\mathcal{F}(U) = S$ for all open sets U. Show that this forms a presheaf (with the obvious restriction maps), and even satisfies the identity axiom. Show that this needn't form a sheaf. (Here we need the axiom that $\mathcal{F}(\emptyset)$ must be the final object, not S. Without this patch, the constant presheaf *is* a sheaf.) This is called the *constant presheaf with values in* S. We will denote this presheaf $\underline{S}^{\text{pre}}$.
(b) Now let $\mathcal{F}(U)$ be the maps to S that are *locally constant*, i.e. for any point x in U, there is a neighborhood of x where the function is constant. (Better description is this: endow S with the discrete topology, and let $\mathcal{F}(U)$ be the continuous maps $U \to S$.) Show that this is a *sheaf*. (Here we need $\mathcal{F}(\emptyset)$ to be the final object again.) We will try to call this the *locally constant sheaf*. (In the real world, this is called the *constant sheaf*. I don't understand why.) We will denote this sheaf $\underline{S}$.

B1 (Yoneda's lemma). Pick an object in your favorite category $A \in \mathcal{C}$. For any object $C \in \mathcal{C}$, we have a set of morphisms $\mathrm{Mor}(C, A)$. If we have a morphism $f : B \to C$, we get a map of sets

$$\mathrm{Mor}(C, A) \to \mathrm{Mor}(B, A), \tag{1}$$

just by composition: given a map from C to A, we immediately get a map from B to A by precomposing with f. Yoneda's lemma, or at least part of it, says that this functor determines A up to unique isomorphism. Translation: If we have two objects A and A', and isomorphisms

$$i_C : \mathrm{Mor}(C, A) \to \mathrm{Mor}(C, A') \tag{2}$$

that commute with the maps (1), then the i_C must be induced from a unique isomorphism $A \to A'$. Prove this.

B2. Prove that a morphism is a monomorphism if and only if the natural morphism $X \to X \times_Y X$ is an isomorphism. (We may then take this as the definition of monomorphism.)

(Monomorphisms aren't very central to future discussions, although they will come up again. This exercise is just good practice.)

B3 (tensor product). (This will be important later!) Suppose $T \to R, S$ are two ring morphisms. Let I be an ideal of R. Let I^e be the extension of I to $R \otimes_T S$. These are the elements $\sum_j i_j \otimes s_j$ where $i_j \in I$, $s_j \in S$. Show that there is a natural isomorphism

$$R/I \otimes_T S \cong (R \otimes_T S)/I^e.$$

Hence the natural morphism $R \otimes_T S \to R/I \otimes_T S$ is a surjection. As an application, we can compute tensor products of finitely generated k algebras over k. For example,

$$k[x_1, x_2]/(x_1^2 - x_2) \otimes_k k[y_1, y_2]/(y_1^3 + y_2^3) \cong k[x_1, x_2, y_1, y_2]/(x_1^2 - x_2, y_1^3 + y_2^3).$$

B4 (direct limits). We say a partially ordered set I is a *directed set* if for $i, j \in I$, there is some $k \in I$ with $i, j \leq k$. In this exercise, you will show that the direct limit of any system of A-modules indexed by I exists, by constructing it. Say the system is given by M_i ($i \in I$), and $f_{ij} : M_i \to M_j$ ($i \leq j$ in I). Let $M = \oplus_i M_i$, where each M_i is identified with its image in M, and let R be the submodule generated by all elements of the form $m_i - f_{ij}(m_i)$ where $m_i \in M_i$ and $i \leq j$. Show that M/R (with the projection maps from the M_i) is $\lim_{\rightarrow} M_i$. You will notice that the same argument works in other interesting categories, such as: sets; groups; and abelian groups. (This example came up in interpreting/defining stalks as direct limits.)

B5 (practice with universal properties). The purpose of this exercise is to give you some practice with "adjoints of forgetful functors", the means by which we get groups from semigroups, and sheaves from presheaves. Suppose R is a ring, and S is a multiplicative subset. Then $S^{-1}R$-modules are a fully faithful subcategory of the category of R-modules (meaning: the objects of the first category are a subset of the objects of the second; and the morphisms between any two objects of the second that are secretly objects of the first are just the morphisms from the first). Then $M \to S^{-1}M$ satisfies a universal property. Figure out what the universal property is, and check that it holds. In other words, describe the universal property enjoyed by $M \to S^{-1}M$, and prove that it holds.

(Here is the larger story. Let $S^{-1}R$-Mod be the category of $S^{-1}R$-modules, and R-Mod be the category of R-modules. Every $S^{-1}R$-module is an R-module, so we have a (covariant) forgetful functor $F : S^{-1}R\text{-Mod} \to R\text{-Mod}$. In fact this is a fully faithful functor: it is injective on objects, and the morphisms between any two $S^{-1}R$-modules *as R-modules* are just the same when they are considered as $S^{-1}R$-modules. Then there is a functor $G : R\text{-Mod} \to S^{-1}R\text{-Mod}$, which might reasonably be called "localization with respect to S", which is left-adjoint to the forgetful functor. Translation: If A is an R-module, and B is an $S^{-1}R$-module, then $\mathrm{Hom}(GA, B)$ (morphisms as $S^{-1}R$-modules, which is incidentally the same as morphisms as R-modules) are in natural bijection with $\mathrm{Hom}(A, FB)$ (morphisms as R-modules).)

B6 (good examples of sheaves). Suppose Y is a topological space. Show that "continuous maps to Y" form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps to Y. Show that this forms a sheaf.

(b) Suppose we are given a continuous map $f : Y \to X$. Show that "sections of f" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps s to Y such that $f \circ s = \mathrm{id}|_U$. Show that this forms a sheaf. (A classical construction of sheaves in general is to interpret them in precisely this way. See Serre's revolutionary article *Faisceaux Algébriques Cohérents*.)

B7 (an important construction, the pushforward sheaf). (a) Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X. Then define $f_*\mathcal{F}$ by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$, where V is an open subset of Y. Show that $f_*\mathcal{F}$ is a sheaf. This is called a *pushforward sheaf*. More precisely, $f_*\mathcal{F}$ is called the *pushforward of $\mathcal{F}$ by f*.
(b) Assume $\mathcal{F}$ is a sheaf of sets (or rings or R-modules), so stalks exist. If $f(x) = y$, describe the natural morphism of stalks $(f_*\mathcal{F})_y \to \mathcal{F}_x$.

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