FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 53 AND 54

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CONTENTS

1. SERRE DUALITY

Our last topic is Serre duality. Recall that Serre duality arose in our section on "fun with curves" (classes 33–36). We'll prove the statement used there, and generalize it greatly.

Our goal is to rigorously prove everything we needed for curves, and to generalize the statement significantly. Serre duality can be generalized beyond belief, and we'll content ourselves with the version that is most useful. For the generalization, we will need a few facts that we haven't proved, but that we came close to proving.

(i) The existence (and behavior) of the cup product in (Cech) cohomology. For any quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$, there is a natural map $H^i(X, \mathcal{F}) \times H^j(X, \mathcal{G}) \to H^{i+j}(X, \mathcal{F} \otimes \mathcal{G})$ satisfying all the properties you might hope. From the Cech cohomology point of view this isn't hard. For those of you who prefer derived functors, I haven't thought through why it is true. For $i = 0$ or $j = 0$, the meaning of the cup product is easy. (For example, if $i = 0$, the map involves the following. The j-cocycle of $\mathcal{G}$ is the data of sections of $\mathcal{G}$ of $(j+1)$-fold intersections of affine open sets. The cup product corresponds to "multiplying

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each of these by the (restriction of the) global section of $\mathcal{F}''$.) This version is all we'll need for nonsingular projective curves (as if $i, j > 0$, $i + j > 1$).

(ii) The Cohen-Macaulay/flatness theorem. I never properly defined Cohen-Macaulay, so I didn't have a chance to prove that nonsingular schemes are Cohen-Macaulay, and if $\pi : X \to Y$ is a morphism from a pure-dimensional Cohen-Macaulay scheme to a pure-dimensional nonsingular scheme, then π is flat if all the fibers are of the expected dimension. (I stated this, however.)

We'll take these two facts for granted.

Here now is the statement of Serre duality.

Suppose X is a Cohen-Macaulay projective k-scheme of pure dimension n. A *dualizing sheaf* for X over k is a coherent sheaf ω_X (or $\omega_{X/k}$) on X along with a *trace map* $H^n(X, \omega_W) \to k$, such that for all finite rank locally free sheaves $\mathcal{F}$ on X,

$$H^i(X, \mathcal{F}) \times H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k \tag{1}$$

is a perfect pairing. In terms of the cup product, the first map in (1) is the composition

$$H^i(X, \mathcal{F}) \times H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X) \to H^n(X, (\mathcal{F} \otimes \mathcal{F}^\vee) \otimes \omega_X) \to H^n(X, \omega_X).$$

1.1. *Theorem (Serre duality). — A dualizing sheaf always exists.*

We will proceed as follows.

- We'll partially extend this to coherent sheaves in general: $\mathrm{Hom}(\mathcal{F}, \omega_X) \to H^n(\mathcal{F})^\vee$ is an isomorphism for all $\mathcal{F}$.
- Using this, we'll show by a Yoneda argument that (ω_X, t) is unique up to unique isomorphism.
- We will then prove the Serre duality theorem 1.1. This will take us some time. We'll first prove that the dualizing sheaf exists for projective space. We'll then prove it for anything admitting a finite flat morphism to projective space. Finally we'll show that every projective Cohen-Macaulay k-scheme admits a finite flat morphism to projective space.
- We'll prove the result in families (i.e. we'll define a "relative dualizing sheaf" in good circumstances). This is useful in the theory of moduli of curves, and Gromov-Witten theory.
- The existence of a dualizing sheaf will be straightforward to show — surprisingly so, at least to me. However, it is also surprisingly slippery — getting a hold of it in concrete circumstances is quite difficult. For example, on the open subset where X is smooth, ω_X is an invertible sheaf. We'll show this. Furthermore, on this locus, $\omega_X = \det \Omega_X$. (Thus in the case of curves, $\omega_X = \Omega_X$. In the "fun with curves" section, we needed the fact that Ω_X is dualizing because we wanted to prove the Riemann-Hurwitz formula.)

1.2. *Warm-up trivial exercise.* Show that if $h^0(X, \mathcal{O}_X) = 1$ (e.g. if X is geometrically integral), then the trace map is an isomorphism, and conversely.

2. EXTENSION TO COHERENT SHEAVES; UNIQUENESS OF THE DUALIZING SHEAF

2.1. *Proposition. — If (ω_X, t) exists, then for any coherent sheaf $\mathcal{F}$ on X, the natural map $\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \to k$ is a perfect pairing.*

In other words, (1) holds for $i = n$ and any coherent sheaf (not just locally free coherent sheaves). You might reasonably ask if it holds for general i, and it is true that these other cases are very useful, although not as useful as the case we're proving here. In fact the naive generalization does not hold. The correct generalization involves Ext groups, which we have not defined. The precise statement is the following. For any quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$, there is a natural map $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \times H^j(X, \mathcal{F}) \to H^{i+j}(\mathcal{G})$. Via this morphism,

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k$$

is a perfect pairing.

Proof of Proposition 2.1. Given any coherent $\mathcal{F}$, take a partial locally free resolution

$$\mathcal{E}^1 \to \mathcal{E}^0 \to \mathcal{F} \to 0.$$

(Recall that we find a locally free resolution as follows. $\mathcal{E}^0$ is a direct sum of line bundles. We then find $\mathcal{E}^1$ that is also a direct sum of line bundles that surjects onto the kernel of $\mathcal{E}^0 \to \mathcal{F}$.)

Then applying the left-exact functor $\mathrm{Hom}(\cdot, \omega_X)$, we get

$$0 \to \mathrm{Hom}(\mathcal{F}, \omega_X) \to \mathrm{Hom}(\mathcal{E}^0, \omega_X) \to \mathrm{Hom}(\mathcal{E}^1, \omega_X)$$
$$\text{i.e.} \quad 0 \to \mathrm{Hom}(\mathcal{F}, \omega_X) \to (\mathcal{E}^0)^\vee \otimes \omega_X \to (\mathcal{E}^1)^\vee \otimes \omega_X$$

Also

$$H^n(\mathcal{E}^1) \to H^n(\mathcal{E}^0) \to H^n(\mathcal{F}) \to 0$$

from which

$$0 \to H^n(\mathcal{F})^\vee \to H^n(\mathcal{E}^0)^\vee \to H^n(\mathcal{E}^1)^\vee$$

There is a natural map $\mathrm{Hom}(\mathcal{H}, \omega_X) \times H^n(\mathcal{H}) \to H^n(\omega_X) \to k$ for all coherent sheaves, which by assumption (that ω_X is dualizing) is an isomorphism when $\mathcal{H}$ is locally free. Thus we have morphisms (where all squares are commuting)

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(\mathcal{F}, \omega) & \longrightarrow & (\mathcal{E}^0)^\vee(\omega) & \longrightarrow & (\mathcal{E}^1)^\vee(\omega) \\
\downarrow\sim & & \downarrow & & \downarrow\sim & & \downarrow\sim \\
0 & \longrightarrow & H^n(\mathcal{F})^\vee & \longrightarrow & H^n(\mathcal{E}^0)^\vee & \longrightarrow & H^n(\mathcal{E}^1)^\vee
\end{array}$$

where all vertical maps but one are known to be isomorphisms. Hence by the Five Lemma, the remaining map is also an isomorphism. □

We can now use Yoneda's lemma to prove:

2.2. *Proposition.* — If a dualizing sheaf (ω_X, t) exists, it is unique up to unique isomorphism.

Proof. Suppose we have two dualizing sheaves, (ω_X, t) and (ω'_X, t'). From the two morphisms

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k \tag{2}$$

$$\mathrm{Hom}(\mathcal{F}, \omega'_X) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega'_X) \xrightarrow{t'} k,$$

we get a natural bijection $\mathrm{Hom}(\mathcal{F}, \omega_X) \cong \mathrm{Hom}(\mathcal{F}, \omega'_X)$, which is functorial in $\mathcal{F}$. By Yoneda's lemma, this induces a (unique) isomorphism $\omega_X \cong \omega'_X$. From (2), under this isomorphism, the two trace maps must be the same too. □

3. Proving Serre duality for projective space over a field

3.1. *Exercise.* Prove (1) for $\mathbb{P}^n$, and $\mathcal{F} = \mathcal{O}(m)$, where $\omega_{\mathbb{P}^n} = \mathcal{O}(-n-1)$. (Hint: do this by hand!) Hence (1) holds for direct sums of $\mathcal{O}(m)$'s.

3.2. *Proposition.* — *Serre duality (Theorem 1.1) holds for projective space.*

Proof. We now prove (1) for any locally free $\mathcal{F}$ on $\mathbb{P}^n$. As usual, take

$$0 \to \mathcal{K} \to \oplus\mathcal{O}(m) \to \mathcal{F} \to 0. \tag{3}$$

Note that $\mathcal{K}$ is flat (as $\mathcal{O}(m)$ and $\mathcal{F}$ are flat and coherent), and hence $\mathcal{K}$ is also locally free of finite rank (flat coherent sheaves on locally Noetherian schemes are locally free — this was one of the important facts about flatness). For convenience, set $\mathcal{G} = \oplus\mathcal{O}(m)$.

Take the long exact sequence in cohomology, and dualize, to obtain

$$0 \to H^n(\mathbb{P}^n, \mathcal{F})^\vee \to H^n(\mathbb{P}^n, \mathcal{G})^\vee \to \cdots \to H^0(\mathbb{P}^n, \mathcal{H})^\vee \to 0. \tag{4}$$

Now instead take (3), tensor with $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$ (which preserves exactness, as $\mathcal{O}_{\mathbb{P}^n}(-n-1)$ is locally free), and take the corresponding long exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) \longrightarrow H^0(\mathbb{P}^n, \mathcal{G}^\vee \otimes \omega_{\mathbb{P}^n}) \longrightarrow H^0(\mathbb{P}^n, \mathcal{H}^\vee \otimes \omega_{\mathbb{P}^n})$$

$$\longrightarrow H^1(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) \longrightarrow \cdots$$

Using the trace morphism, this exact sequence maps to the earlier one (4):

$$\begin{array}{ccccccc}
H^i(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{G}^\vee \otimes \omega_{\mathbb{P}^n}) & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{H}^\vee \otimes \omega_{\mathbb{P}^n}) & \longrightarrow & H^{i+1}(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) \\
\downarrow{\scriptstyle \alpha^i_{\mathcal{F}}} & & \downarrow{\scriptstyle \alpha^i_{\mathcal{G}}} & & \downarrow{\scriptstyle \alpha^i_{\mathcal{H}}} & & \downarrow{\scriptstyle \alpha^{i+1}_{\mathcal{F}}} \\
H^{n-i}(\mathbb{P}^n, \mathcal{F})^\vee & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{G})^\vee & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{H})^\vee & \longrightarrow & H^{i+1}(\mathbb{P}^n, \mathcal{F})^\vee
\end{array}$$

(At some point around here, I could simplify matters by pointing out that $H^i(\mathcal{G}) = 0$ for all $i \neq 0, n$, as $\mathcal{G}$ is the direct sum of line bundles, but then I'd still need to deal with the ends, so I'll prefer not to.) All squares here commute. This is fairly straightforward check for those not involving the connecting homomorphism. (*Exercise.* Check this.) It is longer and more tedious (but equally straightforward) to check that

$$\begin{array}{ccc}
H^i(\mathbb{P}^n, \mathcal{H}^\vee \otimes \omega_{\mathbb{P}^n}) & \longrightarrow & H^{i+1}(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) \\
\downarrow{\scriptstyle \alpha^i_{\mathcal{H}}} & & \downarrow{\scriptstyle \alpha^{i+1}_{\mathcal{F}}} \\
H^i(\mathbb{P}^n, \mathcal{H})^\vee & \longrightarrow & H^{i+1}(\mathbb{P}^n, \mathcal{F})^\vee
\end{array}$$

commutes. This requires the definition of the cup product, which we haven't done, so this is one of the arguments I promised to omit.

We then induct our way through the sequence as usual: $\alpha^{-1}_{\mathcal{G}}$ is surjective (vacuously), and $\alpha^{-1}_{\mathcal{H}}$ and $\alpha^0_{\mathcal{G}}$ are injective, hence by the "subtle" Five Lemma (class 32, page 10), $\alpha^0_{\mathcal{F}}$ is injective for all locally free $\mathcal{F}$. In particular, $\alpha^0_{\mathcal{H}}$ is injective (as $\mathcal{H}$ is locally free). But then $\alpha^0_{\mathcal{H}}$ is injective, and $\alpha^{-1}_{\mathcal{H}}$ and $\alpha^0_{\mathcal{G}}$ are surjective, hence $\alpha^0_{\mathcal{F}}$ is surjective, and thus an isomorphism for all locally free $\mathcal{F}$. Thus $\alpha^0_{\mathcal{H}}$ is also an isomorphism, and we continue inductively to show that $\alpha^i_{\mathcal{F}}$ is an isomorphism for all i. □

4. Proving Serre duality for finite flat covers of other spaces for which duality holds

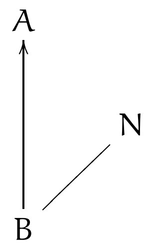
We're now going to make a new construction. It will be relatively elementary to describe, but the intuition is very deep. (Caution: here "cover" doesn't mean covering space as in differential geometry; it just means "surjective map". The word "cover" is often used in this imprecise way in algebraic geometry.)

Suppose $\pi : X \to Y$ is an *affine* morphism, and $\mathcal{G}$ is a quasicoherent sheaf on Y:

$$\begin{array}{ccc}
X & & \\
\downarrow{\scriptstyle \pi} & & \mathcal{G} \\
 & \diagup & \\
Y. & &
\end{array}$$

Observe that $\underline{\mathrm{Hom}}_Y(\pi_* \mathcal{O}_X, \mathcal{G})$ is a sheaf of $\pi_* \mathcal{O}_X$-modules. (The subscript Y is included to remind us where the sheaf lives.) The reason is that affine-locally on Y, over an affine open set $\operatorname{Spec} B$ (on which $\mathcal{G}$ corresponds to B-module N, and with preimage $\operatorname{Spec} A \subset X$)

(5)



this is the statement that $\operatorname{Hom}_B(A, N)$ is naturally an A-module (i.e. the A-module structure behaves well with respect to localization by $b \in B$, and hence these modules glue together to form a quasicoherent sheaf).

In our earlier discussion of affine morphisms, we saw that quasicoherent $\pi_* \mathcal{O}_X$-modules correspond to quasicoherent sheaves on X. Hence $\underline{\operatorname{Hom}}_Y(\pi_* \mathcal{O}_X, \mathcal{G})$ corresponds to some quasicoherent sheaf $\pi' \mathcal{G}$ on X.

Notational warning. This notation π' is my own, and solely for the purposes of this section. If π is finite, then this construction is called $\pi^!$ (pronounced "upper shriek"). You may ask why I'm introducing this extra notation "upper shriek". That's because this notation is standard, while my π' notation is just made up. $\pi^!$ is one of the "six operations" on sheaves defined Grothendieck. It is the most complicated one, and is complicated to define for general π. Those of you attending Young-Hoon Kiem's lectures on the derived category may be a little perplexed, as there he defined $\pi^!$ for elements of the derived category of sheaves, not for sheaves themselves. In the finite case, we can define this notion at the level of sheaves, but we can't in general.

Here are some important observations about this notion.

4.1. By construction, we have an isomorphism of quasicoherent sheaves on Y

$$\pi_* \pi' \mathcal{G} \cong \underline{\operatorname{Hom}}_Y(\pi_* \mathcal{O}_X, \mathcal{G}).$$

4.2. π' is a covariant functor from the category of quasicoherent sheaves on Y to quasicoherent sheaves on X.

4.3. If π is a finite morphism, and Y (and hence X) is locally Noetherian, then π' is a covariant functor from the category of *coherent* sheaves on Y to *coherent* sheaves on X. We show this affine locally, see (5). As A and N are both coherent B-modules, $\operatorname{Hom}_B(A, N)$ is a coherent B-module, hence a finitely generated B-module, and hence a finitely generated A-module, hence a coherent A-module.

4.4. If $\mathcal{F}$ is a quasicoherent sheaf on X, then there is a natural map

$$\pi_* \underline{\operatorname{Hom}}_X(\mathcal{F}, \pi' \mathcal{G}) \to \underline{\operatorname{Hom}}_Y(\pi_* \mathcal{F}, \mathcal{G}). \tag{6}$$

Reason: if M is an A-module, we have a natural map

$$\operatorname{Hom}_A(M, \operatorname{Hom}_B(A, N)) \to \operatorname{Hom}_B(M, N) \tag{7}$$

defined as follows. Given $m \in M$, and an element of $\mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N))$, send m to $\phi_m(1)$. This is clearly a homomorphism of B-modules. Moreover, this morphism behaves well with respect to localization of B with respect to an element of B, and hence this description yields a morphism of quasicoherent sheaves.

4.5. *Lemma. The morphism* (6) *is an isomorphism.*

Is there an obvious reason why the map is an isomorphism? There should be...

Proof. We will show that the natural map (7) is an isomorphism. Fix a presentation of M:

$$A^{\oplus m} \to A^{\oplus n} \to M \to 0$$

(where the direct sums needn't be finite). Applying $\mathrm{Hom}_A(\cdot, \mathrm{Hom}_B(A, N))$ to this sequence yields the top row of the following diagram, and applying $\mathrm{Hom}_B(\cdot, N)$ yields the bottom row, and the vertical morphisms arise from the morphism (7).

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N)) & \longrightarrow & \mathrm{Hom}_A(A, \mathrm{Hom}_B(A, N))^{\oplus n} & \longrightarrow & \mathrm{Hom}_A(A, \mathrm{Hom}_B(A, N))^{\oplus m} \\ \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ 0 & \longrightarrow & \mathrm{Hom}_B(M, N) & \longrightarrow & \mathrm{Hom}_B(A, N)^{\oplus n} & \longrightarrow & \mathrm{Hom}_B(A, N)^{\oplus m} \end{array}$$

(The squares clearly commute.) Be sure to convince yourself that

$$\mathrm{Hom}_B(A, N)^{\oplus n} \cong \mathrm{Hom}_B(A^{\oplus n}, N)$$

even when n isn't finite (and ditto for the three similar terms)! Then all but one of the vertical homomorphisms are isomorphisms, and hence by the Five Lemma the remaining morphism is an isomorphism. $\square$

Hence π' is right-adjoint to π_* for affine morphisms and quasicoherent sheaves. (Also, by Observation 4.3, it is right-adjoint for finite morphisms and coherent sheaves on locally Noetherian schemes.) In particular, there is a natural morphism $\pi_*\pi^!\mathcal{G} \to \mathcal{G}$.

4.6. *Proposition. — Suppose $X \to Y$ is a finite flat morphism of projective k-schemes of pure dimension n, and (ω_Y, t_Y) is a dualizing sheaf for Y. Then $\pi^!\omega_Y$ along with the trace morphism*

$$t_X : H^n(X, \pi^!\omega_Y) \xrightarrow{\sim} H^n(Y, \pi_*\pi^!\omega_Y) \longrightarrow H^n(Y, \omega_Y) \xrightarrow{t_Y} k$$

is a dualizing sheaf for X.

(That first isomorphism arises because $X \to Y$ is affine.)

Proof.

$$\begin{aligned} H^{n-i}(X, \mathcal{F}^\vee(\pi^! \omega_Y)) &\cong H^{n-i}(Y, \pi_*(\mathcal{F}^\vee \otimes \pi^! \omega_Y)) \quad \text{as } \pi \text{ is affine} \\ &\cong H^{n-i}(Y, \pi_*(\underline{\mathrm{Hom}}(\mathcal{F}, \pi^! \omega_Y))) \\ &\cong H^{n-i}(Y, \underline{\mathrm{Hom}}(\pi_* \mathcal{F}, \omega_Y)) \quad \text{by 4.5} \\ &\cong H^{n-i}(Y, (\pi_* \mathcal{F})^\vee(\omega_Y)) \\ &\cong H^i(Y, \pi_* \mathcal{F})^\vee \quad \text{by Serre duality for } Y \\ &\cong H^i(X, \mathcal{F})^\vee \quad \text{as } \pi \text{ is affine} \end{aligned}$$

At the third-last and second-last steps, we are using the fact that $\pi_* \mathcal{F}$ is locally free, and it is here that we are using flatness! □

5. All projective Cohen-Macaulay k-schemes of pure dimension n are finite flat covers of $\mathbb{P}^n$

We conclude the proof of the Serre duality theorem 1.1 by establishing the result in the title of this section.

Assume $X \hookrightarrow \mathbb{P}^N$ is projective Cohen-Macaulay of pure dimension n (e.g. smooth).

First assume that k is an infinite field. Then long ago in an exercise that I promised would be important (and has repeatedly been so), we showed that there is a linear space of dimension $N - n - 1$ (one less than complementary dimension) missing X. Project from that linear space, to obtain $\pi : X \to \mathbb{P}^n$. Note that the fibers are finite (the fibers are all closed subschemes of affine space), and hence π is a finite morphism. I've stated the "Cohen-Macaulay/flatness theorem" that a morphism from a equidimensional Cohen-Macaulay scheme to a smooth k-scheme is flat if and only if the fibers are of the expected dimension. Hence π is flat.

5.1. *Exercise.* Prove the result in general, if k is not necessarily infinite. Hint: show that there is some d such that there is an intersection of $N - n - 1$ degree d hypersurfaces missing X. Then try the above argument with the dth Veronese of $\mathbb{P}^N$.

6. Serre duality in families

6.1. *Exercise: Serre duality in families.* Suppose $\pi : X \to Y$ is a flat projective morphism of locally Noetherian schemes, of relative dimension n. Assume all of the geometric fibers are Cohen-Macaulay. Then there exists a coherent sheaf $\omega_{X/Y}$ on X, along with a trace map $R^n \pi_* \omega_{X/Y} \to \mathcal{O}_Y$ such that, for every finite rank locally free sheaves $\mathcal{F}$ on X, each of whose higher pushforwards are locally free on Y,

$$R^i \pi_* \mathcal{F} \times R^{n-i} \pi_*(\mathcal{F}^\vee \otimes \omega_X) \longrightarrow R^n \pi_* \omega_X \xrightarrow{t} \mathcal{O}_Y \tag{8}$$

is a perfect pairing. (Hint: follow through the same argument!)

Note that the hypothesis, that all higher pushforwards are locally free on Y, is the sort of thing provided by the cohomology and base change theorem. (In the solution to Exercise 6.1, you will likely show that $R^{n-i}\pi_*(\mathcal{F}^\vee \otimes \omega_X)$ is a locally free sheaf for all $\mathcal{F}$ such that $R^i\pi_*\mathcal{F}$ is a locally free sheaf.)

You will need the *fibral flatness theorem* (EGA IV(3).11.3.10–11), which you should feel free to use: if $g : X \to S$, $h : Y \to S$ are locally of finite presentation, and $f : X \to Y$ is an S-morphism, then the following are equivalent:

(a) g is flat and $f_s : X_s \to Y_s$ is flat for all $s \in s$,
(b) h is flat at all points of $f(X)$ and f is flat.

7. What we still want

There are three or four more facts I want you to know.

- On the locus of X where k is smooth, there is an isomorphism $\omega_{X/k} \cong \det \Omega_{X/k}$. (Note for experts: it isn't canonical!) We define $\det \Omega_{X/k}$ to be $\mathcal{K}_X$. We used this in the case of smooth curves over k (proper, geometrically integral). This is surprisingly hard, certainly harder than the mere existence of the canonical sheaf!
- *The adjunction formula.* If D is a Cartier divisor on X (so D is also Cohen-Macaulay, by one of the facts about Cohen-Macaulayness I've mentioned), then $\omega_{D/k} = (\omega_{X/k} \otimes \mathcal{O}_X(D))\,|_D$.

On can show this using Ext groups, but I haven't established their existence or properties. So instead, I'm going to go as far as I can without using them, and then I'll tell you a little about them.

But first, here are some exercises *assuming* that ω is isomorphic to $\det \Omega$ on the smooth locus.

7.1. *Exercise (Serre duality gives a symmetry of the Hodge diamond).* Suppose X is a smooth projective k-variety of dimension n. Define $\Omega_X^p = \wedge^p \Omega X$. Show that we have a natural isomorphism $H^q(X, \Omega^p) \cong H^{n-q}(X, \Omega^{n-p})^\vee$.

7.2. *Exercise (adjunction for smooth subvarieties of smooth varieties).* Suppose X is a smooth projective k-scheme, and and D is a smooth effective Cartier divisor. Show that $\mathcal{K}_D \cong \mathcal{K}_X(D)|_D$. Hence if we knew that $\mathcal{K}_X \cong \omega_X$ and $\mathcal{K}_D \cong \omega_D$, this would let us compute ω_D in terms of ω_X. We will use this shortly.

7.3. *Exercise.* Compute $\mathcal{K}$ for a smooth complete intersection in $\mathbb{P}^N$ of hypersurfaces of degree $d_1, \dots, d_n$. Compute ω for a complete intersection in $\mathbb{P}^N$ of hypersurfaces of degree $d_1, \dots, d_n$. (This will be the same calculation!) Find all possible cases where $\mathcal{K} \cong \mathcal{O}$. These are examples of *Calabi-Yau varieties* (or *Calabi-Yau manifolds* if $k = \mathbb{C}$), at least when they have dimension at least 2. If they have dimension precisely 2, they are called K3 surfaces.

8. THE DUALIZING SHEAF IS AN INVERTIBLE SHEAF ON THE SMOOTH LOCUS

(I didn't do this in class, but promised it in the notes. A simpler proof in the case where X is a curve is given in §9.)

We begin with some preliminaries.

(0) If $f : U \to U$ is the identity, and $\mathcal{F}$ is a quasicoherent sheaf on U, then $f'\mathcal{F} \cong \mathcal{F}$.

(i) The $'$ construction behaves well with respect to flat base change, as the pushforward does. In other words, if

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle e} \\ Y' & \xrightarrow{f} & Y \end{array}$$

is a fiber diagram, where f (and hence h) is flat, and $\mathcal{F}$ is any quasicoherent sheaf on Y, then there is a canonical isomorphism $h^*e'\mathcal{F} \cong g'f^*\mathcal{F}$.

(ii) The $'$ construction behaves well with respect to disjoint unions of the source. In other words, if $f_i : X_i \to Y$ $(i = 1, 2)$ are two morphisms, $f : X_1 \cup X_2 \to Y$ is the induced morphism from the disjoint union, and $\mathcal{F}$ is a quasicoherent sheaf on Y, then $f'\mathcal{F}$ is $f_1'\mathcal{F}$ on X_1 and $f_2'\mathcal{F}$ on X_2. The reason again is that pushforward behaves well with respect to disjoint union.

Exercise. Prove both these facts, using abstract nonsense.

Given a smooth point $x \in X$, we can choose our projection so that $\pi : X \to \mathbb{P}^n$ is etale at that point. *Exercise.* Prove this. (Hint: We need only check isomorphisms of tangent spaces.)

So hence we need only check our desired result on the etale locus U for $X \to \mathbb{P}^n$. (This is an open set, as etaleness is an open condition.) Consider the base change.

$$\begin{array}{ccc} X \times_{\mathbb{P}^n_k} U & \xrightarrow{h} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle e} \\ U & \xrightarrow{f} & \mathbb{P}^n_k. \end{array}$$

There is a section $U \to X \times_{\mathbb{P}^n} U$ of the vertical morphism on the left. *Exercise.* Show that it expresses U as a connected component of $X \times_{\mathbb{P}^n} U$. (Hint: Show that a section of an etale morphism always expresses the target as a component of the source as follows. Check that s is a homeomorphism onto its image. Use Nakayama's lemma.) The dualizing sheaf $\omega_{\mathbb{P}^n_k}$ is invertible, and hence $f^*\omega_{\mathbb{P}^n_k}$ is invertible on U. Hence $g^! f^*\omega_{\mathbb{P}^n_k}$ is invertible on $s(U)$ (by observation (0)). By observation (i) then, $h^*g^*\omega_{\mathbb{P}^n_k} \cong h^*\omega_X$ is an invertible sheaf.

We are now reduced to showing the following. Suppose $h : U \to X$ is an etale morphism. (In the etale topology, this is called an "etale open set", even though it isn't an open set in any reasonable sense.) Its image is an open subset of X (as etale morphisms are open maps). Suppose $\mathcal{F}$ is a coherent sheaf on X such that $h^*\mathcal{F}$ is an invertible sheaf on U. Then $\mathcal{F}$ is an invertible sheaf on the image of U.

(Experts will notice that this is a special case of *faithfully flat descent*.)

Exercise. Prove this. Hint: it suffices to check that the stalks of $\mathcal{F}$ are isomorphic to the stalks of the structure sheaf. Hence reduce the question to a map of local rings: suppose $(B, \mathfrak{n}) \to (A, \mathfrak{m})$ is etale, and N is a coherent B-module such that $M := N \otimes_B A$ is isomorphic to A. We wish to show that N is isomorphic to B. Use Nakayama's lemma to show that N has the same minimal number of generators (over B) as M (over A), by showing that $\dim_{B/\mathfrak{n}} N - \dim_{A/\mathfrak{m}} M$. Hence this number is 1, so $N \cong B/I$ for some ideal I. Then show that $I = 0$ — you'll use flatness here.

9. An easier proof that the dualizing sheaf of a smooth curve is invertible

Here is another proof that for curves, the dualizing sheaf is invertible. We'll show that it is torsion-free, and rank 1.

First, here is why it is rank 1 at the generic point. We have observed that $f^!$ behaves well with respect to flat base change. Suppose L/K is a finite extension of degree n. Then $\mathrm{Hom}_K(L, K)$ is an L-module. What is its rank? As a K-module, it has rank n. Hence as an L-module it has rank 1. Applying this to $C \to \mathbb{P}^1$ at the generic point ($L = FF(C)$, $K = FF(\mathbb{P}^1)$) gives us the desired result. (Side remark: its structure as an L-module is a little mysterious. You can see that some sort of duality is relevant here. Illuminating this module's structure involves the norm map.)

Conclusion: the dualizing sheaf is rank 1 at the generic point.

Here is why it is torsion free. Let ω_t be the torsion part of ω, and ω_{nt} be the torsion-free part, so we have an exact sequence

$$0 \to \omega_t \to \omega \to \omega_{nt} \to 0.$$

9.1. *Exercise.* Show that this splits: $\omega = \omega_t \oplus \omega_{nt}$. (Hint: It suffices to find a splitting map $\omega \to \omega_t$. As ω_t is supported at a finite set of points, it suffices to find this map in a neighborhood of one of the points in the support. Restrict to a small enough affine open set where ω_{nt} is free. Then on this there is a splitting $\omega_{nt} \to \omega$, from which on that open set we have a splitting $\omega \to \omega_t$.)

Notice that ω_{nt} is rank 1 and torsion-free, hence an invertible sheaf. By Serre duality, for any invertible sheaf $\mathcal{L}$, $h^0(\mathcal{L}) = h^1(\omega_{nt} \otimes \mathcal{L}^\vee)$ and $h^1(\mathcal{L}) = h^0(\omega_{nt} \otimes \mathcal{L}^\vee) + h^0(\omega_t \otimes \mathcal{L})$. Substitute $\mathcal{L} = \mathcal{O}_X$ in the first of these equations and $\mathcal{L} = \omega_X$ in the second, to obtain that $h^0(X, \omega_t) = 0$. But the only skyscraper sheaf with no sections is the 0 sheaf, hence $\omega_t = 0$.

One can show that the determinant of the sheaf of differentials is the dualizing sheaf using Ext groups, but this involves developing some more machinery, without proof. Instead, I'd like to prove it directly for curves, using what we already have proved. (Note again that our proof of Serre duality for curves was rigorous — the cup product was already well-defined for dimension 1 schemes.)

I'll do this in a sequence of exercises.

Suppose C is an geometrically irreducible, smooth projective k-curve.

We wish to show that $\Omega_C \cong \omega_C$. Both are invertible sheaves. (Proofs that ω_C is invertible were given in §8 and §9.)

Define the genus of a curve as $g = h^1(C, \mathcal{O}_C)$. By Serre duality, this is $h^0(C, \omega_C)$. Also, $h^0(C, \mathcal{O}_C) = h^1(C, \omega_C) = 1$.

Suppose we knew that $h^0(C, \Omega_C) = h^0(C, \omega_C)$, and $h^1(C, \Omega_C) = h^1(\omega_C)$ $(= 1)$. Then $\deg \Omega_C = \deg \omega_C$. Also, by Serre duality $h^0(C, \Omega_C^\vee \otimes \omega_C) = h^1(\Omega_C) = 1$. Thus $\Omega_C^\vee \otimes \omega_C$ is a degree 0 invertible sheaf with a nonzero section. We have seen that this implies that the sheaf is trivial, so $\Omega_C \cong \omega_C$.

Thus it suffices to prove that $h^1(C, \Omega_C) = 1$, and $h^0(C, \Omega_C) = h^0(C, \omega_C)$. By Serre duality, we can restate the latter equality without reference to ω: $h^0(C, \Omega) = h^1(C, \mathcal{O}_C)$. Note that we can assume $k = \overline{k}$: all three cohomology group dimensions $h^i(C, \Omega_C)$, $h^0(C, \mathcal{O}_C)$ are preserved by field extension (shown earlier).

Until this point, the argument is slick and direct. What remains is reasonably pleasant, but circuitous. Can you think of a faster way to proceed, for example using branched covers of $\mathbb{P}^1$?

10.1. *Exercise.* Show that C can be expressed as a plane curve with only nodes as singularities. (Hint: embed C in a large projective space, and take a general projection. The Kleiman-Bertini theorem, or at least its method of proof, will be handy.)

Let the degree of this plane curve be d, and the number of nodes be δ. We then blow up $\mathbb{P}^2$ at the nodes (let $S = \operatorname{Bl} \mathbb{P}^2$), obtaining a closed immersion $C \hookrightarrow S$. Let H be the divisor class that is the pullback of the line ($\mathcal{O}(1)$) on $\mathbb{P}^2$. Let $E_1, \ldots, E_\delta$ be the classes of the exceptional divisors.

10.2. *Exercise.* Show that the class of C on $\mathbb{P}^2$ is $dH - 2\sum E_i$. (Reason: the total transform has class dH. Each exceptional divisor appears in the total transform with multiplicity two.)

10.3. *Exercise.* Use long exact sequences to show that $h^1(C, \mathcal{O}_C) = \binom{d-1}{2} - \delta$. (Hint: Compute $\chi(C, \mathcal{O}_C)$ instead. One possibility is to compute $\chi(C', \mathcal{O}_{C'})$ where C' is the image of C in $\mathbb{P}^2$, and use the Leray spectral sequence for $C \to C'$. Another possibility is to work on S directly.)

10.4. *Exercise.* Show that $\Omega_C = \mathcal{K}_S(C)|_C$. Show that this is

$$(-3H + \sum E_i) + (dH - \sum 2E_i).$$

Show that this has degree $2g - 2$ where $g = h^1(\mathcal{O}_C)$. (Possible hint: use long exact sequences.)

10.5. *Exercise.* Show that $h^0(\Omega_C) > 2g - 2 - g + 1 = g - 1$ from

$$0 \to H^0(S, \mathcal{K}_S) \to H^0(S, \mathcal{K}_S(C)) \to H^0(C, \Omega_C).$$

10.6. *Exercise.* Show that $\Omega_C \cong \omega_C$.

11. Ext groups, and adjunction

Let me now introduce Ext groups and their properties, without proof. Suppose i is a non-negative integer. Given two quasicoherent sheaves, $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ is a quasicoherent sheaf. $\mathrm{Ext}^0(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. Then there are long exact sequences in both arguments. In other words, if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a short exact sequence, then there is a long exact sequence starting

$$0 \to \mathrm{Ext}^0(\mathcal{F}'', \mathcal{G}) \to \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^0(\mathcal{F}', \mathcal{G}) \to \mathrm{Ext}^1(\mathcal{F}'', \mathcal{G}) \to \cdots,$$

and if

$$0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$$

is a short exact sequence, then there is a long exact sequence starting

$$0 \to \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}') \to \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}'') \to \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}') \to \cdots.$$

Also, if $\mathcal{F}$ is locally free, there is a canonical isomorphism $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \cong H^i(X, \mathcal{G} \otimes \mathcal{F}^\vee)$.

For any quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$, there is a natural map $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \times H^j(X, \mathcal{F}) \to H^{i+j}(\mathcal{G})$.

For any coherent sheaf on X, there is a natural morphism ("cup product") $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \times H^j(X, \mathcal{F}) \to H^{i+j}(X, \mathcal{G})$.

11.1. *Exercise.* Suppose X is Cohen-Macaulay, and finite type and projective over k (so Serre duality holds). Via this morphism, show that

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k$$

is a perfect pairing. Feel free to assume whatever nice properties of Ext-groups you need (as we haven't proven any of them anyway).

Hence Serre duality yields a natural extension to coherent sheaves. This is sometimes called Serre duality as well. This more general statement is handy to prove the adjunction formula.

11.2. *Adjunction formula.* — If X is a Serre duality space (i.e. a space where Serre duality holds), and D is an effective Cartier divisor, then $\omega_D = (\omega_X \otimes \mathcal{O}(D))|_D$.

We've seen that if X and D were smooth, and we knew that $\omega_X \cong \det \Omega_X$ and $\omega_D \cong \det \Omega_D$, we would be able to prove this easily (Exercise 7.2).

But we get more. For example, complete intersections in projective space have invertible dualizing sheaves, no matter how singular or how nonreduced. Indeed, complete intersections in *any* smooth projective k-scheme have invertible dualizing sheaves.

A projective k-schemes with invertible dualizing sheaf is so nice that it has a name: it is said to be *Gorenstein*. (Gorenstein has a more general definition, that also involves a dualizing sheaf. It is a local definition, like nonsingularity and Cohen-Macaulayness.)

11.3. *Exercise.* Prove the adjunction formula. (Hint: Consider $0 \to \omega_X \to \omega_X(D) \to \omega_X(D)|_D \to 0$. Apply $\mathrm{Hom}_X(\mathcal{F}, \cdot)$ to this, and take the long exact sequence in Ext-groups.) As before, feel free to assume whatever facts about Ext groups you need.

The following exercise is a bit distasteful, but potentially handy. Most likely you should skip it, and just show that $\omega_X \cong \det \Omega_X$ using the theory of Ext groups.

11.4. *Exercise.* We make a (temporary) definition inductively by definition. A k-variety is "nice" if it is smooth, and (i) it has dimension 0 or 1, or (ii) for any nontrivial invertible sheaf $\mathcal{L}$ on X, there is a nice divisor D such that $\mathcal{L}|_D \neq 0$. Show that for any nice k-variety, $\omega_X \cong \det \Omega_X$. (Hint: use the adjunction formula, and the fact that we know the result for curves.)

11.5. *Remark.* You may wonder if ω_X is always an invertible sheaf. In fact it isn't, for example if $X = \operatorname{Spec} k[x,y]/(x,y)^2$. I think I can give you a neat and short explanation of this fact. If you are curious, just ask.

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