FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 49 AND 50

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CONTENTS

At the start of class 49, I gave an informal discussion on other criteria for ampleness, and other adjectives for divisors. We discussed the following notions: Kleiman's criterion for ampleness, numerical equivalence, Neron-Severi group, Picard number, nef, the nef cone and the ample cone, Nakai's criterion, the Nakai-Moishezon criterion, big, $\mathbb{Q}$-Cartier, Snapper's theorem.)

1. Blowing up a scheme along a closed subscheme

We'll next discuss an important construction in algebraic geometry (and especially the geometric side of the subject), the blow-up of a scheme along a closed subscheme (cut out by a finite type ideal sheaf). We'll start with a motivational example that will give you a picture of the construction in a particularly important case (and the historically earliest case), in Section 2. I'll then give a formal definition, in terms of universal property, Section 3. This definition won't immediately have a clear connection to the motivational example! We'll deduce some consequences of the definition (assuming that the blow-up actually exists). We'll prove that the blow-up always exists, by describing it quite explicitly, in Section 4. As a consequence, the blow-up morphism is projective, and we'll deduce more consequences from this. In Section 5, we'll do a number of explicit computations, and see that in practice, it is possible to compute many things by hand. I'll then mention a couple of useful facts: (i) the blow-up a nonsingular variety in a nonsingular variety is still nonsingular, something we'll have observed in our explicit examples, and (ii) Castelnuovo's criterion, that on a smooth surface, "(-1)-curves" ($\mathbb{P}^1$'s with normal bundle $\mathcal{O}(-1)$) can be "blown down".

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2. MOTIVATIONAL EXAMPLE

We're going to generalize the following notion, which will correspond to "blowing up" the origin of $\mathbb{A}^2_k$ (over an algebraically closed field k). Because this is just motivation, I'll be informal. Consider the subset of $\mathbb{A}^2 \times \mathbb{P}^1$ corresponding to the following. We interpret $\mathbb{P}^1$ as the lines through the origin. Consider the subset $\{(p \in \mathbb{A}^2, [\ell] \in \mathbb{P}^1) : p \in \ell)\}$. (I showed you a model in class, admittedly over the non-algebraically-closed field $k = \mathbb{R}$.)

I'll now try to convince you that this is nonsingular (informally). Now $\mathbb{P}^1$ is smooth, and for each point $[\ell]$ in $\mathbb{P}^1$, we have a smooth choice of points on the line ℓ. Thus we are verifying smoothness by way of the fibration over $\mathbb{P}^1$.

Let's make this more algebraic. Let x and y be coordinates on $\mathbb{A}^2$, and X and Y be projective coordinates on $\mathbb{P}^1$ ("corresponding" to x and y); we'll consider the subset $\operatorname{Bl}_{(0,0)} \mathbb{A}^2$ of $\mathbb{A}^2 \times \mathbb{P}^1$ corresponding to $xY - yX = 0$. We could then verify that this is nonsingular (by looking at two covering patches).

Notice that the preimage of $(0,0)$ is a curve and hence a divisor (an effective Cartier divisor, as the blown-up surface is nonsingular). Also, note that if we have some curve singular at the origin, this could be partially desingularized. (A *desingularization* or a *resolution of singularities* of a variety X is a proper birational morphism $\tilde{X} \to X$ from a nonsingular scheme. We are interested in desingularizations for many reasons. For example, we understand nonsingular curves quite well. and we could hope to understand other curves through their desingularizations. This philosophy holds true in higher dimension as well.) For example, the curve $y^2 = x^3 + x^2$, which is nonsingular except for a node at the origin, then we can take the preimage of the curve minus the origin, and take the closure of this locus in the blow-up, and we'll obtain a nonsingular curve; the two branches of the node downstairs are separated upstairs. (This will later be an exercise, once we've defined things properly. The result will be called the *proper transform* of the curve.)

Let's generalize this. First, we can blow up $\mathbb{A}^n$ at the origin (or more informally, "blow up the origin"), getting a subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$. More algebraically, If $x_1, \ldots, x_n$ are coordinates on $\mathbb{A}^n$, and $X_1, \ldots, X_n$ are projective coordinates on $\mathbb{P}^{n-1}$, then the blow-up $\operatorname{Bl}_{\vec{0}} \mathbb{A}^n$ is given by the equations $x_i X_j - x_j X_i = 0$. Once again, this is smooth: $\mathbb{P}^{n-1}$ is smooth, and for each point $[\ell] \in \mathbb{P}^{n-1}$, we have a smooth choice of $p \in \ell$.

We can extend this further, by blowing up $\mathbb{A}^{n+m}$ along a coordinate m-plane $\mathbb{A}^n$ by adding m more variables $x_{n+1}, \ldots, x_{n+m}$ to the previous example; we get a subset of $\mathbb{A}^{n+m} \times \mathbb{P}^{n-1}$.

Then intuitively, we could extend this to blowing up a nonsingular subvariety of a nonsingular variety. We'll make this more precise. In the course of doing so, we will accidentally generalize this notion greatly, defining the blow-up of any finite type sheaf of ideals in a scheme. In general, blowing up may not have such an intuitive description as in the case of blowing up something nonsingular inside something nonsingular — it does great violence to the scheme — but even then, it is very useful (for example, in developing intersection theory). The result will be very powerful, and will touch on many other useful notions in algebra (such as the Rees algebra) that we won't discuss here.

Our description will depend only the closed subscheme being blown up, and not on coordinates. That remedies a defect was already present in the first baby example, blowing up the plane at the origin. It is not obvious that if we picked different coordinates for the plane (preserving the origin as a closed subscheme) that we wouldn't have two different resulting blow-ups.

As is often the case, there are two ways of understanding this notion, and each is useful in different circumstances. The first is by universal property, which lets you show some things without any work. The second is an explicit construction, which lets you get your hands dirty and compute things (and implies for example that the blow-up morphism is projective).

3. Blowing up, by universal property

I'll start by defining the blow-up using the universal property. The disadvantage of starting here is that this definition won't obviously be the same as the examples I just gave. It won't even look related!

Suppose $X \hookrightarrow Y$ is a closed subscheme corresponding to a finite type sheaf of ideals. (If Y is locally Noetherian, the "finite type" hypothesis is automatic, so Noetherian readers can ignore it.)

The blow-up $X \hookrightarrow Y$ is a *fiber diagram*

$$\begin{tikzcd} E_X Y \arrow[r, hook] \arrow[d] & \mathrm{Bl}_X Y \arrow[d, "\beta"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

such that $E_X Y$ is an *effective Cartier divisor* on $\mathrm{Bl}_X Y$ (and is the scheme-theoretical pullback of X on Y), such any other such fiber diagram

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & W \arrow[d] \\ X \arrow[r, hook] & Y, \end{tikzcd} \tag{1}$$

where D is an effective Cartier divisor on W, factors uniquely through it:

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & W \arrow[d] \\ E_X Y \arrow[r, hook] \arrow[d] & \mathrm{Bl}_X Y \arrow[d] \\ X \arrow[r, hook] & Y. \end{tikzcd}$$

(Recall that an effective Cartier divisor is locally cut out by one equation that is not a zero-divisor; equivalently, it is locally cut out by one equation, and contains no associated points. This latter description will prove crucial.) $\mathrm{Bl}_X Y$ is called *the blow-up* (of Y along X, or of Y with center X). $E_X Y$ is called the *exceptional divisor*. (Bl and β stand for "blow-up", and E stands for "exceptional".)

By a universal property argument, if the blow-up exists, it is unique up to unique isomorphism. (We can even recast this more explicitly in the language of Yoneda's lemma: consider the category of diagrams of the form (1), where morphisms are of the form

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & W \arrow[d] \\ D' \arrow[r, hook] \arrow[d] & W' \arrow[d] \\ X \arrow[r, hook] & Y. \end{tikzcd}$$

Then the blow-up is a final object in this category, if one exists.)

If $Z \hookrightarrow Y$ is any closed subscheme of Y, then the (scheme-theoretic) pullback $\beta^{-1}Z$ is called the *total transform* of Z. We will soon see that β is an isomorphism away from X (Observation 3.4). $\overline{\beta^{-1}(Z - X)}$ is called the *proper transform* or *strict transform* of Z. (We will use the first terminology. We will also define it in a more general situation.) We'll soon see that the proper transform is naturally isomorphic to $\mathrm{Bl}_{Z\cap X} Z$, where by $Z \cap X$ we mean the scheme-theoretic intersection (the blow-up closure lemma 3.7).

We will soon show that the blow-up always exists, and describe it explicitly. But first, we make a series of observations, assuming that the blow up exists.

3.1. *Observation.* If X is the empty set, then $\mathrm{Bl}_X Y = Y$. More generally, if X is a Cartier divisor, then the blow-up is an isomorphism. (Reason: $\mathrm{id}_Y : Y \to Y$ satisfies the universal property.)

3.2. *Exercise.* If U is an open subset of Y, then $\mathrm{Bl}_{U\cap X} U \cong \beta^{-1}(U)$, where $\beta : \mathrm{Bl}_X Y \to Y$ is the blow-up. (Hint: show $\beta^{-1}(U)$ satisfies the universal property!)

Thus "we can compute the blow-up locally."

3.3. *Exercise.* Show that if Y_α is an open cover of Y (as α runs over some index set), and the blow-up of Y_α along $X \cap Y_\alpha$ exists, then the blow-up of Y along X exists.

3.4. *Observation.* Combining Observation 3.1 and Exercise 3.2, we see that the blow-up is an isomorphism away from the locus you are blowing up:

$$\beta|_{\mathrm{Bl}_X Y - E_X Y} : \mathrm{Bl}_X Y - E_X Y \to Y - X$$

is an isomorphism.

3.5. *Observation.* If $X = Y$, then the blow-up is the empty set: the only map $W \to Y$ such that the pullback of X is a Cartier divisor is $\emptyset \hookrightarrow Y$. In this case we have "blown Y out of existence"!

3.6. *Exercise (blow-up preserves irreducibility and reducedness).* Show that if Y is irreducible, and X doesn't contain the generic point of Y, then $\mathrm{Bl}_X Y$ is irreducible. Show that if Y is reduced, then $\mathrm{Bl}_X Y$ is reduced.

The following blow-up closure lemma is useful in several ways. At first, it is confusing to look at, but once you look closely you'll realize that it is not so unreasonable.

Suppose we have a fibered diagram

$$\begin{tikzcd} W \arrow[r, hook, "\text{cl. imm.}"] \arrow[d] & Z \arrow[d] \\ X \arrow[r, hook, "\text{cl. imm.}"] & Y \end{tikzcd}$$

where the bottom closed immersion corresponds to a finite type ideal sheaf (and hence the upper closed immersion does too). The first time you read this, it may be helpful to consider the special case where $Z \to Y$ is a closed immersion.

Then take the fiber product of this square by the blow-up $\beta : \mathrm{Bl}_X Y \to Y$, to obtain

$$\begin{tikzcd} Z \times_Y E_X Y \arrow[r, hook] \arrow[d] & Z \times_Y \mathrm{Bl}_X Y \arrow[d] \\ E_X Y \arrow[r, hook, "\text{Cartier}"] & \mathrm{Bl}_X Y. \end{tikzcd}$$

The bottom closed immersion is locally cut out by one equation, and thus the same is true of the top closed immersion as well. However, it need not be a non-zero-divisor, and thus the top closed immersion is not necessarily an effective Cartier divisor.

Let $\overline{Z}$ be the scheme-theoretic closure of $Z \times_Y \mathrm{Bl}_X Y - W \times_Y \mathrm{Bl}_X Y$ in $Z \times_Y \mathrm{Bl}_X Y$. Note that in the special case where $Z \to Y$ is a closed immersion, $\overline{Z}$ is the proper transform, as defined in §3. For this reason, it is reasonable to call $\overline{Z}$ the proper transform of Z even if Z *isn't* a closed immersion. Similarly, it is reasonable to call $Z \times_Z \mathrm{Bl}_X Y$ the total transform even if Z isn't a closed immersion.

Define $E_{\overline{Z}} \hookrightarrow \overline{Z}$ as the pullback of $E_X Y$ to $\overline{Z}$, i.e. by the fibered diagram

$$\begin{tikzcd} E_{\overline{Z}} \arrow[r, hook] \arrow[d, hook, "\text{cl. imm.}"] & \overline{Z} \arrow[d, hook, "\text{cl. imm.}"] & \text{proper transform} \\ Z \times_Y E_X Y \arrow[r, hook] \arrow[d] & Z \times_Y \mathrm{Bl}_X Y \arrow[d] & \text{total transform} \\ E_X Y \arrow[r, hook, "\text{Cartier}"] & \mathrm{Bl}_X Y. & \end{tikzcd}$$

Note that $E_{\overline{Z}}$ is Cartier on $\overline{Z}$ (as it is locally the zero-scheme of a single function that does not vanish on any associated points of $\overline{Z}$).

3.7. *Blow-up closure lemma.* — $(\mathrm{Bl}_Z W, E_Z W)$ *is canonically isomorphic to* $(\overline{Z}, E_{\overline{Z}})$.

This is very handy.

The first three comments apply to the special case where $Z \to W$ is a closed immersion, and the fourth basically tells us we shouldn't have concentrated on this special case.

(1) First, note that if $Z \to Y$ is a closed immersion, then this states that the proper transform (as defined in §3) is the blow-up of Z along the scheme-theoretic intersection $W = X \cap Z$.

(2) In particular, it lets you actually compute blow-ups, and we'll do lots of examples soon. For example, suppose C is a plane curve, singular at a point p, and we want to blow up C at p. Then we could instead blow up the plane at p (which we have already described how to do, even if we haven't yet proved that it satisfies the universal property of blowing up), and then take the scheme-theoretic closure of $C - p$ in the blow-up.

(3) More generally, if W is some nasty subscheme of Z that we wanted to blow-up, and Z were a finite type k-scheme, then the same trick would work. We could work locally (Exercise 3.2), so we may assume that Z is affine. If W is cut out by r equations $f_1, \dots, f_r \in \Gamma(\mathcal{O}_Z)$, then complete the f's to a generating set $f_1, \dots, f_n$ of $\Gamma(\mathcal{O}_Z)$. This gives a closed immersion $Y \hookrightarrow \mathbb{A}^n$ such that W is the scheme-theoretic intersection of Y with a coordinate linear space $\mathbb{A}^r$.

3.8. (4) Most generally still, this reduces the existence of the blow-up to a specific special case. (If you prefer to work over a fixed field k, feel free to replace $\mathbb{Z}$ by k in this discussion.) Suppose that for each n, $\mathrm{Bl}_{(x_1,\dots,x_n)} \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$ exists. Then I claim that the blow-up always exists. Here's why. We may assume that Y is affine, say $\operatorname{Spec} B$, and $X = \operatorname{Spec} B/(f_1, \dots, f_n)$. Then we have a morphism $Y \to \mathbb{A}^n_{\mathbb{Z}}$ given by $x_i \mapsto f_i$, such that X is the scheme-theoretic pullback of the origin. Hence by the blow-up closure lemma, $\mathrm{Bl}_X Y$ exists.

3.9. *Tricky Exercise+.* Prove the blow-up closure lemma. Hint: obviously, construct maps in both directions, using the universal property. The following diagram may or may not help.

$$
\begin{tikzcd}
 & & & & E_{\overline{Z}} \arrow[rr, hook, "\text{Cartier}"] \arrow[dd, hook, "\text{cl. imm.}"] \arrow[ddllll] & & \overline{Z} \arrow[dd, hook, "\text{cl. imm.}"] \arrow[ddllll] \arrow[dllll, dotted] \\
E_W Z \arrow[rr, hook, "\text{Cartier}"] \arrow[d] \arrow[urrrr, dotted] & & \mathrm{Bl}_W Z \arrow[d] & & & & \\
W \arrow[rr, hook] \arrow[drr] & & Z \arrow[drr] & & E_X Y \arrow[rr, hook, "\text{Cartier}"] \arrow[dll] & & \mathrm{Bl}_X Y \arrow[dll] \\
 & & X \arrow[rr, hook] & & Y & &
\end{tikzcd}
$$

3.10. *Exercise.* If Y and Z are closed subschemes of a given scheme X, show that $\mathrm{Bl}_Y Y \cup Z \cong \mathrm{Bl}_{Y\cap Z} Z$. (In particular, if you blow up a scheme along an irreducible component, the irreducible component is blown out of existence.)

4. The blow-up exists, and is projective

It is now time to show that the blow up always exists. I'll give two arguments, because I find them enlightening in two different ways. Both will imply that the blow-up morphism is projective. Hence the blow-up morphism is projective, hence quasicompact, proper, finite type, separated. In particular, if $Y \to Z$ is projective (resp. quasiprojective, quasicompact, proper, finite type, separated), so is $\mathrm{Bl}_X Y \to Z$. The blow-up of a k-variety is a k-variety (using the fact that irreducibility, reducedness are preserved, Exercise 3.6).

Approach 1. As explained above (§3.8), it suffices to show that $\mathrm{Bl}_{(x_1,\dots,x_n)} \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$ exists. But we know what it is supposed to be: the locus in

$$\operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n] \times \operatorname{Proj} \mathbb{Z}[X_1, \dots, X_n]$$

such that $x_i X_j - x_j X_i = 0$. We'll show this soon.

Approach 2. We can describe the blow-up all at once as a $\underline{\mathrm{Proj}}$.

4.1. *Theorem ($\underline{\mathrm{Proj}}$ description of the blow-up). — Suppose $X \hookrightarrow Y$ is a closed subscheme cut out by a finite type sheaf of ideals $\mathcal{I} \hookrightarrow \mathcal{O}_Y$. Then*

$$\underline{\mathrm{Proj}}\left(\mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots\right) \to Y$$

satisfies the universal property of blowing up.

We'll prove this soon (Section 4.2), after seeing what this gives us. (The reason we had a finite type requirement is that I wanted this $\underline{\mathrm{Proj}}$ to exist; we needed the sheaf of algebras to satisfy the conditions stated earlier.)

But first, we should make sure that the preimage of X is indeed an effective Cartier divisor. We can work affine-locally (Exercise 3.2), so I'll assume that $Y = \operatorname{Spec} B$, and X is cut out by the finitely generated ideal I. Then

$$\mathrm{Bl}_X Y = \operatorname{Proj}\left(B \oplus I \oplus I^2 \oplus \cdots\right).$$

(We are slightly abusing notation by using the notation $\mathrm{Bl}_X Y$, as we haven't yet shown that this satisfies the universal property. But I hope that by now you trust me.)

The preimage of X isn't just any effective Cartier divisor; it corresponds to the invertible sheaf $\mathcal{O}(1)$ on this $\underline{\operatorname{Proj}}$. Indeed, $\mathcal{O}(1)$ corresponds to taking our graded ring, chopping off the bottom piece, and sliding all the graded pieces to the left by 1; it is the invertible sheaf corresponding to the graded module

$$I \oplus I^2 \oplus I^3 \oplus \cdots$$

(where that first summand I has grading 0). But this can be interpreted as the scheme-theoretic pullback of X, which corresponds to the ideal I of B:

$$I\left(B \oplus I \oplus I^2 \oplus \cdots\right) \hookrightarrow B \oplus I \oplus I^2 \oplus \cdots.$$

Thus the scheme-theoretic pullback of $X \hookrightarrow Y$ to $\underline{\operatorname{Proj}}\, \mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \cdots$, the invertible sheaf corresponding to $\mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots$, is an effective Cartier divisor in class $\mathcal{O}(1)$. Once we have verified that this construction is indeed the blow-up, this divisor will be our exceptional divisor $E_X Y$.

Moreover, we see that the exceptional divisor can be described beautifully as a $\underline{\operatorname{Proj}}$ over X:

(2) $$E_X Y = \underline{\operatorname{Proj}}_X\, B/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots.$$

We'll later see that in good circumstances (if X is a local complete intersection in something nonsingular, or more generally a local complete intersection in a Cohen-Macaulay scheme) this is a projective bundle (the "projectivized normal bundle").

4.2. Proof of the universal property, Theorem 4.1. Let's prove that this $\underline{\operatorname{Proj}}$ construction satisfies the universal property. Then approach 1 will also follow, as a special case of approach 2. You may ask why I bothered with approach 1. I have two reasons: one is that you may find it more comfortable to work with this one nice ring, and the picture may be geometrically clearer to you (in the same way that thinking about the blow-up closure lemma in the case where $Z \to Y$ is a closed immersion is more intuitive). The second reason is that, as you'll find in the exercises, you'll see some facts more easily in this explicit example, and you can then pull them back to more general examples.

Proof. Reduce to the case of affine target R with ideal I. Reduce to the case of affine source, with principal effective Cartier divisor t. (A principal effective Cartier divisor is cut out by a single non-zero-divisor. Recall that an effective Cartier divisor is cut out only *locally* by a single non-zero divisor.) Thus we have reduced to the case $\operatorname{Spec} S \to \operatorname{Spec} R$, corresponding to $f : R \to S$. Say $(x_1, \ldots, x_n) = I$, with $(f(x_1), \ldots, f(x_n)) = (t)$. We'll describe *one* map $\operatorname{Spec} S \to \operatorname{Proj} R[I]$ that will extend the map on the open set $\operatorname{Spec} S_t \to \operatorname{Spec} R$. It is then unique: a map to a separated R-scheme is determined by its behavior away from the associated points (proved earlier). We map R[I] to S as follows: the degree one part is $f : R \to S$, and $f(X_i)$ (where X_i corresponds to x_i, except it is in degree 1) goes to $f(x_i)/t$. Hence an element X of degree d goes to $X/(t^d)$. On the open set $D_+(X_1)$, we get the map $R[X_2/X_1, \dots, X_n/X_1]/(x_2 - X_2/X_1 x_1, \dots, x_i X_j - x_j X_i, \dots) \to S$ (where there may be many relations) which agrees with f away from $D(t)$. Thus this map does extend away from $V(I)$. $\square$

Here are some applications and observations arising from this construction of the blow-up.

4.3. *Observation.* We can verify that our initial motivational examples are indeed blow-ups. For example, blowing up $\mathbb{A}^2$ (with co-ordinates x and y) at the origin yields: $B = k[x, y]$, $I = (x, y)$, and $\operatorname{Proj} B \oplus I \oplus I^2 = \operatorname{Proj} B[X, Y]$ where the elements of B have degree 0, and X and Y are degree 1 and correspond to x and y.

4.4. *Observation.* Note that the normal bundle to a Cartier divisor D is the invertible sheaf $\mathcal{O}(D)|_D$, the invertible sheaf corresponding to the D on the total space, then restricted to D. (This was discussed earlier in the section on differentials.) (Reason: if D corresponds to the ideal sheaf $\mathcal{I}$, then recall that $\mathcal{I} = \mathcal{O}(D)^\vee$, and that the conormal sheaf was $\mathcal{I}/\mathcal{I}^2 = \mathcal{I}|_D$.) The ideal sheaf corresponding to the exceptional divisor is $\mathcal{O}(1)$, so the invertible sheaf corresponding to the exceptional divisor is $\mathcal{O}(-1)$. (I prefer to think of this in light of approach 1, but there is no real difference.) Thus for example in the case of the blow-up of a point in the plane, the exceptional divisor has normal bundle $\mathcal{O}(-1)$. In the case of the blow-up of a nonsingular subvariety of a nonsingular variety, the blow up turns out to be nonsingular (a fact discussed soon in §6.1), and the exceptional divisor is a projective bundle over X, and the normal bundle to the exceptional divisor restricts to $\mathcal{O}(-1)$.

4.5. More serious application: dimensional vanishing for quasicoherent sheaves on quasiprojective schemes. Here is something promised long ago. I want to point out something interesting here: in proof I give below, we will need to potentially blow up arbitrary closed schemes. We won't need to understand precisely what happens when we do so; all we need is the fact that the exceptional divisor is indeed a (Cartier) divisor.

5. Explicit computations

In this section you will do a number of explicit of examples, to get a sense of how blow-ups behave, how they are useful, and how one can work with them explicitly. For convenience, all of the following are over an algebraically closed field k of characteristic 0.

5.1. *Example: Blowing up the plane along the origin.* Let's first blow up the plane $\mathbb{A}^2_k$ along the origin, and see that the result agrees with our discussion in §2. Let x and y be the coordinates on $\mathbb{A}^2_k$. The the blow-up is $\operatorname{Proj} k[x, y, X, Y]$ where $xY - yX = 0$. This is naturally a closed subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$, cut out (in terms of the projective coordinates X and Y on $\mathbb{P}^1_k$) by $xY - yX = 0$. We consider the two usual patches on $\mathbb{P}^1_k$: $[X; Y] = [s; 1]$ and $[1; t]$. The first patch yields $\operatorname{Spec} k[x, y, s]/(sy - x)$, and the second gives $\operatorname{Spec} k[x, y, t]/(y -$

xt). Notice that both are nonsingular: the first is naturally $\operatorname{Spec} k[y, s] \cong \mathbb{A}_k^2$, the second is $\operatorname{Spec} k[x, t] \cong \mathbb{A}_k^2$.

Let's describe the exceptional divisor. We first consider the first (s) patch. The ideal is generated by (x, y), which in our ys-coordinates is $(ys, y) = (y)$, which is indeed principal. Thus on this patch the exceptional divisor is generated by y. Similarly, in the second patch, the exceptional divisor is cut out by x. (This can be a little confusing, but there is no contradiction!)

5.2. *The proper transform of a nodal curve.* Consider next the curve $y^2 = x^3 + x^2$ inside the plane $\mathbb{A}_k^2$. Let's blow up the origin, and compute the total and proper transform of the curve. (By the blow-up closure lemma, the latter is the blow-up of the nodal curve at the origin.) In the first patch, we get $y^2 - s^2y^2 - s^3y^3 = 0$. This factors: we get the exceptional divisor y with multiplicity two, and the curve $1 - s^2 - y^3 = 0$. Easy exercise: check that the proper transform is nonsingular. Also, notice where the proper transform meets the exceptional divisor: at two points, $s = \pm 1$. This corresponds to the two tangent directions at the origin. (Notice that $s = y/x$.)

5.3. *Exercise.* Describe both the total and proper transform of the curve C given by $y = x^2 - x$ in $\operatorname{Bl}_{(0,0)} \mathbb{A}^2$. Verify that the proper transform of C is isomorphic to C. Interpret the intersection of the proper transform of C with the exceptional divisor E as the slope of C at the origin.

5.4. *Exercise: blowing up a cuspidal plane curve.* Describe the proper transform of the cuspidal curve C' given by $y^2 = x^3$ in the plane $\mathbb{A}_k^2$. Show that it is nonsingular. Show that the proper transform of C meets the exceptional divisor E at one point, and is tangent to E there.

5.5. *Exercise.* (a) Desingularize the tacnode $y^2 = x^4$ by blowing up the plane at the origin (and taking the proper transform), and then blowing up the resulting surface once more.
(b) Desingularize $y^8 - x^5 = 0$ in the same way. How many blow-ups do you need?
(c) Do (a) instead in one step by blowing up (y, x^2).

5.6. *Exercise.* Blowing up a nonreduced subscheme of a nonsingular scheme can give you something singular, as shown in this example. Describe the blow up of the ideal (x, y^2) in $\mathbb{A}_k^2$. What singularity do you get? (Hint: it appears in a nearby exercise.)

5.7. *Exercise.* Blow up the cone point $z^2 = x^2 + y^2$ at the origin. Show that the resulting surface is nonsingular. Show that the exceptional divisor is isomorphic to $\mathbb{P}^1$.

5.8. *Harder but enlightening exercise.* If $X \hookrightarrow \mathbb{P}^n$ is a projective scheme, show that the exceptional divisor of the blow up the affine cone over X at the origin is isomorphic to X, and that its normal bundle is $\mathcal{O}_X(-1)$. (I prefer approach 1 here, but both work.)

In the case $X = \mathbb{P}^1$, we recover the blow-up of the plane at a point. In particular, we again recover the important fact that the normal bundle to the exceptional divisor is $\mathcal{O}(-1)$.

5.9. *Exercise.* Show that the multiplicity of the exceptional divisor in the total transform of a subscheme of $\mathbb{A}^n$ when you blow up the origin is the lowest degree that appears in a defining equation of the subscheme. (For example, in the case of the nodal and cuspidal curves above, Example 5.2 and Exercise 5.4 respectively, the exceptional divisor appears with multiplicity 2.) This is called the *multiplicity* of the singularity.

5.10. *Exercise.* Suppose Y is the cone $x^2 + y^2 = z^2$, and X is the ruling of the cone $x = 0$, $y = z$. Show that $\mathrm{Bl}_X Y$ is nonsingular. (In this case we are blowing up a codimension 1 locus that is not a Cartier divisor. Note that it *is* Cartier away from the cone point, so you should expect your answer to be an isomorphism away from the cone point.)

5.11. *Harder but useful exercise (blow-ups resolve base loci of rational maps to projective space).* (I find this easier via method 1.) Suppose we have a scheme Y, an invertible sheaf $\mathcal{L}$, and a number of sections $s_0, \ldots, s_n$ of $\mathcal{L}$. Then away from the closed subscheme X cut out by $s_0 = \cdots = s_n = 0$, these sections give a morphism to $\mathbb{P}^n$. Show that this morphism extends to a morphism $\mathrm{Bl}_X Y \to \mathbb{P}^n$, where this morphism corresponds to the invertible sheaf $(\pi^*\mathcal{L})(-E_X Y)$, where $\pi : \mathrm{Bl}_X Y \to Y$ is the blow-up morphism. In other words, "blowing up the base scheme resolves this rational map". (Hint: it suffices to consider an affine open subset of Y where $\mathcal{L}$ is trivial.)

6. Two stray facts

There are two stray facts I want to mention.

6.1. Blowing up a nonsingular in a nonsingular. The first is that if you blow up a nonsingular subscheme of a nonsingular locally Noetherian scheme, the result is nonsingular. I didn't have the time to prove this, but I discussed some of the mathematics behind it. (This is harder than our previous discussion. Also, it uses a flavor of argument that in general I haven't gotten to, about local complete intersections and Cohen-Macaulayness.) Moreover, for a local complete intersection $X \hookrightarrow Y$ cut out by ideal sheaf $\mathcal{I}$, I/I^2 is locally free (class 39/40, Theorem 2.20, p. 10). Then it is a fact (unproved here) that for a local complete intersection, the natural map $\mathrm{Sym}^n \mathcal{I}/\mathcal{I}^2 \to \mathcal{I}^n/\mathcal{I}^{n+1}$ is an isomorphism. Of course it suffices to prove this for affine open sets. More generally, if A is Cohen-Macaulay (recall that I've stated that nonsingular schemes are Cohen-Macaulay), and $x_1, \ldots, x_r \in \mathfrak{m}$ is a regular sequence, with $I = (x_1, \ldots, x_r)$, then the natural map is an isomorphism. You can read about this at p. 110 of Matsumura's Commutative Algebra.

Assuming this fact, we conclude that if $X \hookrightarrow Y$ is a complete intersection in a nonsingular scheme (or more generally cut out by a regular sequence in a Cohen-Macaulay scheme), the exceptional divisor is the projectivized normal bundle (by (2)). (Exercise:

Blow up (xy, z) in $\mathbb{A}^3$, and verify that the exceptional divisor is indeed the projectivized normal bundle.)

In particular, in the case where we blow up a nonsingular subvariety in a nonsingular variety, the exceptional divisor is nonsingular. We can then show that the blow-up is nonsingular as follows. The blow-up $\mathrm{Bl}_X Y$ remains nonsingular away from $E_X Y$, as it is here isomorphic to the nonsingular space $Y - X$. Thus we need check only the exceptional divisor. Fix any point of the exceptional divisor p. Then the dimension of $E_X Y$ at p is precisely the dimension of the Zariski tangent space (by nonsingularity). Moreover, the dimension of $\mathrm{Bl}_X Y$ at p is one more than that of $E_X Y$ (by Krull's Principal Ideal Theorem), as the latter is an effective Cartier divisor), and the dimension of the Zariski tangent space of $\mathrm{Bl}_X Y$ at p is at most one more than that of $E_X Y$. But the first of these is at most as big as the second, so we must have equality, which means that $\mathrm{Bl}_X Y$ is nonsingular at p.

6.2. *Exercise.* Suppose X is an irreducible nonsingular subvariety of a nonsingular variety Y, of codimension at least 2. Describe a natural isomorphism $\mathrm{Pic}\,\mathrm{Bl}_X Y \cong \mathrm{Pic}\, Y \oplus \mathbb{Z}$. (Hint: compare divisors on $\mathrm{Bl}_X Y$ and Y. Show that the exceptional divisor $E_X Y$ gives a non-torsion element of $\mathrm{Pic}(\mathrm{Bl}_X Y)$ by describing a $\mathbb{P}^1$ on $\mathrm{Bl}_X Y$ which has intersection number -1 with $E_X Y$.)

(If I had more time, I would have used this to give Hironaka's example of a nonprojective proper nonsingular threefold. If you are curious and have ten minutes, please ask me! It includes our nonprojective proper surface as a closed subscheme, and indeed that is how we can show nonprojectivity.)

6.3. Castelnuovo's criterion.

A curve in a nonsingular surface that is isomorphic to $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1)$ is called a (-1)-curve. We've shown that if we blow up a nonsingular point of a surface at a (reduced) point, the exceptional divisor is a (-1)-curve. Castelnuovo's criterion is the converse: if we have a quasiprojective surface containing a (-1)-curve, that surface is obtained by blowing up another surface at a reduced nonsingular point. (We say that we can "blow down" the (-1)-curve.)

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