FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 43 AND 44

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This week: constancy of Euler characteristic in flat families. The semicontinuity theorem and consequences. Glimpses of the relative Picard scheme.

1. FLAT IMPLIES CONSTANT EULER CHARACTERISTIC

We come to an important consequence of flatness. We'll see that this result implies many answers and examples to questions that we would have asked before we even knew about flatness.

1.1. *Important Theorem. — Suppose* $f : X \to Y$ *is a projective morphism, and* $\mathcal{F}$ *is a coherent sheaf on* X*, flat over* Y*. Suppose* Y *is locally Noetherian. Then* $\sum(-1)^i h^i(X_y \mathcal{F}|_y)$ *is a locally constant function of* $y \in Y$*. In other words, the Euler characteristic of* $\mathcal{F}$ *is constant in the fibers.*

This is first sign that cohomology behaves well in families. (We'll soon see a second: the Semicontinuity Theorem 4.4.) Before getting to the proof, I'll show you some of its many consequences. (A second proof will be given after the semicontinuity discussion.)

The theorem also gives a necessary condition for flatness. It also sufficient if target is integral and locally Noetherian, although we won't use this. (Reference: You can translate Hartshorne Theorem III.9.9 into this.) I seem to recall that both the necessary and sufficient conditions are due to Serre, but I'm not sure of a reference. It is possible that integrality is not necessary, and that reducedness suffices, but I haven't checked.

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1.2. *Corollary. — Assume the same hypotheses and notation as in Theorem 1.1. Then the Hilbert polynomial of $\mathcal{F}$ is locally constant as a function of $y \in Y$.*

Thus for example a flat family of varieties in projective space will all have the same degree and genus (and the same dimension!). Another consequence of the corollary is something remarkably useful.

1.3. *Corollary. — An invertible sheaf on a flat projective family of connected nonsingular curves has locally constant degree on the fibers.*

Proof. An invertible sheaf $\mathcal{L}$ on a flat family of curves is always flat (as locally it is isomorphic to the structure sheaf). Hence $\chi(\mathcal{L}_y)$ is constant. From the Riemann-Roch formula $\chi(\mathcal{L}_y) = \deg(\mathcal{L}_y) - g(X_y) + 1$, using the local constancy of $\chi(\mathcal{L}_y)$, the result follows. $\square$

Riemann-Roch holds in more general circumstances, and hence the corollary does too. Technically, in the example I'm about to give, we need Riemann-Roch for the union of two $\mathbb{P}^1$'s, which I haven't shown. This can be shown in three ways. (i) I'll prove that Riemann-Roch holds for projective generically reduced curves later. (ii) You can prove it by hand, as an exercise. (iii) You can consider this curve C inside $\mathbb{P}^1 \times \mathbb{P}^1$ as the union of a "vertical fiber" and "horizontal fiber". Any invertible sheaf on C is the restriction of some $\mathcal{O}(a, b)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Use additivity of Euler characteristics on $0 \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a-1, b-1) \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b) \to \mathcal{O}_C(a, b) \to 0$, and note that we have earlier computed the $\chi(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(c, d))$.

This result has a lot of interesting consequences.

1.4. *Example of a proper non-projective surface.* We can use it to show that a certain proper surface is not projective. Here is how.

Fix any field with more than two elements. We begin with a flat projective family of curves whose $X \to \mathbb{P}^1$, such that the fiber X_0 over 0 is isomorphic to $\mathbb{P}^1$, and the fiber X_∞ over ∞ is isomorphic to two $\mathbb{P}^1$'s meeting at a point, $X_\infty = Y_\infty \cup Z_\infty$. For example, consider the family of conics in $\mathbb{P}^2$ (with projective coordinates x, y, z) parameterized by $\mathbb{P}^1$ (with projective coordinates λ and μ given by

$$\lambda xy + \mu z(x + y + z) = 0.$$

This family unfortunately is singular for $[\lambda; \mu] = [0; 1]$ (as well as $[1; 0]$ and one other point), so change coordinates on $\mathbb{P}^1$ so that we obtain a family of the desired form.

We now take a break from this example to discuss an occasionally useful construction.

1.5. *Gluing two schemes together along isomorphic closed subschemes.* Suppose X' and X'' are two schemes, with closed subschemes $W' \hookrightarrow X'$ and $W'' \hookrightarrow X''$, and an isomorphism $W' \to W''$. Then we can glue together X' and X'' along $W' \cong W''$. We define this more formally as the *coproduct*:

$$\begin{tikzcd} W' \cong W'' \arrow[r] \arrow[d] & X' \arrow[d] \\ X'' \arrow[r] & ?. \end{tikzcd}$$

Exercise. Prove that this coproduct exists. Possible hint: work by analogy with our product construction. If the coproduct exists, it is unique up to unique isomorphism. Start with judiciously chosen affine open subsets, and glue.

Warning: You might hope that if you have a scheme X with two disjoint closed subschemes W' and W'', and an isomorphism $W' \to W''$, then you should be able to glue X to itself along $W' \to W''$. This is not always possible! I'll give an example shortly. You can still make sense of the quotient as an *algebraic space*, which I will not define here. If you want to know what it is, ask Jarod, or come to one of the three lectures he'll give later this quarter.

1.6. *Back to the non-projective surface.* Now take two copies of the X we defined above; call them X' and X''. Glue X' to X'' by identifying X'_0 with Y''_∞ (in any way you want) and Y'_∞ with X''_0. (Somewhat more explicitly: we are choosing an isomorphism $X'_0 \cup Y'_\infty$ with $X''_0 \cup Y''_\infty$ that "interchanges the components".) I claim that the resulting surface X is proper and not projective over the base field k. The first is an exercise.

Exercise. Show that X is proper over k. (Hint: show that the union of two proper schemes is also proper.)

Suppose now that X is projective, and is embedded in projective space by an invertible sheaf (line bundle) $\mathcal{L}$. Then the degree of $\mathcal{L}$ on each curve of X is non-negative. For any curve $C \subset X$, let $\deg C$ be the degree of $\mathcal{L}$ on C (or equivalently, the degree of C under this projective embedding). Pull $\mathcal{L}$ back to X'. Then this is a line bundle on a flat projective family, so the degree is constant in fibers. Thus

$$\deg X'_0 = \deg(Y'_\infty \cup Z'_\infty) = \deg Y'_\infty + \deg Z'_\infty > \deg Y'_\infty.$$

(Technically, we have not shown that the middle equality holds, so you should think about why that is clear.) Similarly $\deg X''_0 > \deg Y''_\infty$. But after gluing, $X'_0 = Y''_\infty$ and $X''_0 = Y'_\infty$, so we have a contradiction.

1.7. *Remark.* This is a stripped down version of Hironaka's example in dimension 3. Hironaka's example has the advantage of being nonsingular. I'll present that example (and show how this one comes from Hironaka) when we discuss blow-ups. (I think it is a fact that nonsingular proper surfaces over a field are always projective.)

1.8. *Unimportant remark.* You can do more fun things with this example. For example, we know that projective surfaces can be covered by three affine open sets. This can be used to give an example of (for any N) a proper surface that requires at least N affine open subsets to cover it (see my paper with Mike Roth on my preprints page, Example 4.9).

1.9. *Problematic nature of the notion of "projective morphism".* This example shows that the notion of being projective isn't a great notion. There are four possible definitions that might go with this notion. (1) We are following Grothendieck's definition. This notion is not local on the base. For example, by following the gluing above for the morphisms $X' \to \mathbb{P}^1$ and $X'' \to \mathbb{P}^1$, we obtain a morphism $\pi : X \to \mathbb{P}^1 \cup \mathbb{P}^1$, where the union on the right is obtained by gluing the 0 of the first $\mathbb{P}^1$ to the ∞ of the second, and vice versa. Then away from each node of the target, π is projective. (You could even give some explicit equations if you wanted.) However, we know that π is not projective, as $\rho : \mathbb{P}^1 \cup \mathbb{P}^1 \to \operatorname{Spec} k$ is projective, but we have already shown that $\rho \circ \pi : X \to \operatorname{Spec} k$ is not projective.

(2) Hartshorne's definition is designed for finite type k-schemes, and is definitely the wrong one for schemes in general.

(3) You could make our notion "local on the base" by also requiring more information: e.g. the notion of a projective morphism could be a morphism of schemes $X \to Y$ along with an invertible sheaf $\mathcal{L}$ on X that serves as an $\mathcal{O}(1)$. This is a little unpleasant; when someone says "consider a projective surface", they usually wouldn't want to have any particular projective embedding preferred.

(4) Another possible notion is that of *locally projective*: $\pi : X \to Y$ is locally projective if there is an open cover of Y by U_i such that over each U_i, π is projective (in our original sense (1)). The disadvantage is that this isn't closed under composition, as is shown by our example $X \to \mathbb{P}^1 \cup \mathbb{P}^1 \to \operatorname{Spec} k$.

1.10. *Example: You can't always glue a scheme to itself along isomorphic disjoint subschemes.* In class, we had an impromptu discussion of this, so it is a little rough. I'll use a variation of the above example. We'll see that you can't glue X to itself along an isomorphism $X_0 \cong Y_\infty$. (To make this a precise statement: there is no morphism $\pi : X \to W$ such that there is a curve $C \hookrightarrow W$ such that $\pi^{-1}(W - U) = X - X_0 - Y_\infty$, and π maps both X_0 and Y_∞ isomorphically to W.) A picture here is essential!

If there were such a scheme W, consider the point $\pi(Y_\infty \cap Z_\infty) \in W$. It has an affine neighborhood U; let K be its complement. Consider $\pi^{-1}(K)$. This is a closed subset of X, missing $Y_\infty \cap Z_\infty$. Note that it meets Y_∞ (as the affine open U can contain no $\mathbb{P}^1$'s) and Z_∞. Discard all components of $\pi^{-1}(K)$ that are dimension 0, and that contain components of fibers; call what's left K'. *Caution: I need to make sure that I don't end up discarding the points on Y_∞ and Z_∞. I could show that $\pi^{-1}(K)$ has pure codimension 1, but I'd like to avoid doing that. For now, assume that is the case; I may patch this later.* Then K' is an effective Cartier divisor, inducing an invertible sheaf on the surface X, which in turn is a flat projective family over $\mathbb{P}^1$. Thus the degree of K' is constant on fibers. Then we get the same sort of contradiction:

$$\deg_{K'} Y_\infty = \deg_{K'} X_0 = \deg_{K'} Y_\infty + \deg_{K'} Z_\infty > \deg_{K'} Y_\infty.$$

This led to a more wide-ranging discussion. A surprisingly easy theorem (which you can find in Mumford's Abelian Varieties for example) states that if X is a projective k-scheme with an action by a finite group G, then the quotient X/G exists, and is also a projective scheme. (One first has to define what one means by X/G!) If you are a little careful in choosing the isomorphisms used to build our nonprojective surface (picking

$X_0' \to Y_\infty''$ and $X_0'' \to Y_\infty'$ to be the "same" isomorphisms), then there is a $\mathbb{Z}/2$-action on X ("swapping the $\mathbb{P}^1$'s"), we have shown that the quotient W does *not* exist as a scheme, hence giving another proof (modulo things we haven't shown) that X is not projective.

2. Proof of Important Theorem on constancy of Euler characteristic in flat families

Now you've seen a number of interesting results that seem to have nothing to do with flatness. I find this a good motivation for this motivation: using the concept, we can prove things that were interested in beforehand. It is time to finally prove Theorem 1.1.

Proof. The question is local on the base, so we may reduce to case Y is affine, say $Y = \operatorname{Spec} B$, so $X \hookrightarrow \mathbb{P}^n_B$ for some n. We may reduce to the case $X = \mathbb{P}^n_B$ (as we can consider $\mathcal{F}$ as a sheaf on $\mathbb{P}^n_B$). We may reduce to showing that Hilbert polynomial $\mathcal{F}(m)$ is locally constant for all $m \gg 0$ (as by Serre vanishing for $m \gg 0$, the Hilbert polynomial agrees with the Euler characteristic). Now consider the Cech complex $\mathcal{C}^*$ for $\mathcal{F}$. Note that all the terms in the Cech complex are flat. Twist by $\mathcal{O}(m)$ for $m \gg 0$, so that all the higher pushforwards vanish. Hence $\Gamma(\mathcal{C}^*(m))$ is exact except at the first term, where the cohomology is $\Gamma(\pi_*\mathcal{F}(m))$. We tack on this module to the front of the complex, so it is once again exact. Thus (by an earlier exercise), as we have an exact sequence in which all but the first terms are known to be flat, the first term is flat as well. As it is finitely generated, it is also free by an earlier fact (flat and finitely generated over a Noetherian local ring equals free), and thus has constant rank.

We're interested in the cohomology of the fibers. To obtain that, we tensor the Cech resolution with $k(y)$ (as y runs over Y) and take cohomology. Now the extended Cech resolution (with $\Gamma(\pi_*\mathcal{F}(m))$ tacked on the front) is an exact sequence of flat modules, and hence remains exact upon tensoring with $k(y)$ (or indeed anything else). (Useful translation: cohomology commutes with base change.) Thus $\Gamma(\pi_*\mathcal{F}(m)) \otimes k(y) \cong \Gamma(\pi_*\mathcal{F}(m)|_y)$. Thus the dimension of the Hilbert function is the rank of the locally free sheaf at that point, which is locally constant. $\square$

3. Start of Thursday's class: Review

At this point, you've already seen a large number of facts about flatness. Don't be overwhelmed by them; keep in mind that you care about this concept because we have answered questions we cared about even before knowing about flatness. Here are three examples. (i) If you have a short exact sequence where the last is locally free, then you can tensor with anything and the exact sequence will remain exact. (ii) We described a morphism that is proper but not projective. (iii) We showed that you can't always glue a scheme to itself.

Here is a summary of what we know, highlighting the hard things.

- definition; basic properties (pullback and localization). flat base change commutes with higher pushforwards
- Tor: definition and symmetry. (Hence tensor exact sequences of flats with anything and keep exactness.)
- ideal-theoretic criterion: $\mathrm{Tor}_1(M, A/I) = 0$ for all I. (flatness over PID = torsion-free; over dual numbers) (important special case: DVR)
- for coherent modules over Noetherian local rings, flat=locally free
- flatness is open in good circumstances (flat + lft of lN is open; we should need only weaker hypotheses)
- euler characteristics behave well in projective flat families. In particular, the degree of an invertible sheaf on a flat projective family of curves is locally constant.

4. Cohomology and base change theorems

Here is the type of question we are considering. We'd like to see how higher pushforwards behave with respect to base change. For example, we've seen that higher pushforward commutes with *flat* base change. A special case of base change is the inclusion of a point, so this question specializes to the question: can you tell the cohomology of the fiber from the higher pushforward? The next group of theorems I'll discussed deal with this issue. I'll prove things for projective morphisms. The statements are true for proper morphisms of Noetherian schemes too; the one fact you'll see that I need is the following: that the higher direct image sheaves of coherent sheaves under proper morphisms are also coherent. (I'm largely following Mumford's *Abelian Varieties*. The geometrically interesting theorems all flow from the following neat but unmotivated result.

4.1. *Key theorem. — Suppose* $\pi : X \to \operatorname{Spec} B$ *is a projective morphism of Noetherian [needed?] schemes, and* $\mathcal{F}$ *is a coherent sheaf on* X*, flat over* $\operatorname{Spec} B$*. Then there is a* finite *complex*

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finitely generated projective B*-modules and an isomorphism of functors*

$$H^p(X \times_B A, \mathcal{F} \otimes_B A) \cong H^p(K^* \otimes_B A) \tag{1}$$

for all $p \geq 0$ *in the category of* B*-algebras* A*.*

In fact, K^i will be free for $i > 0$. For $i = 0$, it is projective hence flat hence locally free (by an earlier theorem) on Y.

Translation/idea: Given $\pi : X \to \operatorname{Spec} B$, we will have a complex of vector bundles on the target that computes cohomology (higher-pushforwards), "universally" (even after any base change). The idea is as follows: take the Cech complex, produce a "quasiisomorphic" complex (a complex with the same cohomology) of free modules. For those taking derived category class: we have an isomorphic object in the derived category which is easier to deal with as a complex. We'll first construct the complex so that (1) holds for $B = A$, and then show the result for general A later. Let's put this into practice.

4.2. *Lemma. — Let* C^* *be a complex of* B*-modules such that* $H^i(C^*)$ *are finitely generated* B*-modules, and that* $C^p \neq 0$ *only if* $0 \le p \le n$*. Then there exists a complex* K^* *of finitely generated* B*-modules such that* $K^p \neq 0$ *only if* $0 \le p \le n$ *and* K^p *is free for* $p \ge 1$*, and a homomorphism of complexes* $\phi : K^* \to C^*$ *such that* ϕ *induces isomorphisms* $H^i(K^*) \to H^i(C^*)$ *for all* i*.*

Note that K^i *is* B*-flat for* $i > 0$*. Moreover, if* C^p *are* B*-flat, then* K^0 *is* B*-flat too.*

For all of our purposes except for a side remark, I'd prefer a cleaner statement, where C^* is a complex of B-modules, with $C^p \neq 0$ only if $p \le n$ (in other words, there could be infinitely many non-zero C^p's). The proof is then about half as long

Proof. Step 1. We'll build this complex inductively, and worry about K^0 when we get there.

$$\begin{tikzcd} & & K^m \arrow[r, "\delta^m"] \arrow[d, "\phi_m"] & K^{m+1} \arrow[r, "\delta^{m+1}"] \arrow[d, "\phi_{m+1}"] & K^{m+2} \arrow[r] \arrow[d, "\phi_{m+2}"] & \cdots \\ \cdots \arrow[r] & C^{m-1} \arrow[r] & C^m \arrow[r, "\delta^m"'] & C^{m+1} \arrow[r, "\delta^{m+1}"'] & C^{m+2} \arrow[r] & \cdots . \end{tikzcd}$$

We assume we've defined (K^p, ϕ_p, δ^p) for $p \ge m+1$ such that these squares commute, and the top row is a complex, and ϕ^p defines an isomorphism of cohomology $H^q(K^*) \to H^q(C^*)$ for $q \ge m+2$ and a surjection $\ker \delta^{m+1} \to H^{m+1}(C^*)$, and the K^p are finitely generated B-modules.

We'll adjust the complex to make ϕ_{m+1} an isomorphism of cohomology, and then again to make ϕ_m a surjection on cohomology. Let $B^{m+1} = \ker(\delta^{m+1} : H^{m+1}(K^*) \to H^{m+1}(C^*))$. Then we choose generators, and make these K_1^m. We have a new complex. We get the 0-maps on cohomology at level m. We then add more in to surject on cohomology on level m.

Now what happens when we get to $m = 0$? We have maps of complexes, where everything in the top row is free, and we have an isomorphism of cohomology everywhere except for K^0, where we have a surjection of cohomology. Replace K^0 by $K^0 / \ker \delta^0 \cap \ker \phi_0$. Then this gives an isomorphism of cohomology.

Step 2. We need to check that K^0 is B-flat. Note that everything else in this quasiisomorphism is B-flat. Here is a clever trick: construct the *mapping cylinder* (call it M^*):

$$0 \to K^0 \to C^0 \oplus K^1 \to C^1 \oplus K^2 \to \cdots \to C^{n-1} \oplus K^n \to C^n \to 0.$$

Then we have a short exact sequence of complexes

$$0 \to C^* \to M^* \to K^*[1] \to 0$$

(where $K^*[1]$ is just the same complex as K^*, except slid over by one) yielding isomorphisms of cohomology $H^*(K^*) \to H^*(C^*)$, from which $H^*(M^*) = 0$. (This was an earlier exercise: given a map of complexes induces an isomorphism on cohomology, the mapping cylinder is exact.) Now look back at the mapping cylinder M^*, which we now realize is an exact sequence. All terms in it are flat except possibly K^0. Hence K^0 is flat too (also by an earlier exercise)! $\square$

4.3. *Lemma. — Suppose* $K^* \to C^*$ *is a morphism of finite complexes of* **flat** B*-modules inducing isomorphisms of cohomology (a "quasiisomorphism"). Then for every* B*-algebra* A*, the maps* $H^p(C^* \otimes_B A) \to H^p(K^* \otimes_B A)$ *are isomorphisms.*

Proof. Consider the mapping cylinder M^*, which we know is exact. Then $M^* \otimes_B A$ is still exact! (The reason was our earlier exercise that any exact sequence of flat modules tensored with anything remains flat.) But $M^* \otimes_B A$ is the mapping cylinder of $K^* \otimes_B A \to C^* \otimes_B A$, so this is a quasiisomorphism too. □

Now let's prove the theorem!

Proof of theorem 4.1. Choose a finite covering (e.g. the standard covering). Take the Cech complex C^* for $\mathcal{F}$. Apply the first lemma to get the nicer version K^* of the same complex C^*. Apply the second lemma to see that if you tensor with B and take cohomology, you get the same answer whether you use K^* or C^*. □

We are now ready to put this into use. We will use it to discuss a trio of facts: the Semicontinuity Theorem, Grauert's Theorem, and the Cohomology and Base Change Theorem. (We'll prove the first two.) The theorem of constancy of euler characteristic in flat families also fits in this family.

These theorems involve the following situation. Suppose $\mathcal{F}$ is a coherent sheaf on X, $\pi : X \to Y$ projective, Y (hence X) Noetherian, and $\mathcal{F}$ flat over Y.

Here are two related questions. Is $R^p\pi_*\mathcal{F}$ locally free? Is $\phi^p : R^p\pi_*\mathcal{F}\otimes k(y) \to H^p(X_y, \mathcal{F}_y)$ an isomorphism?

We have shown Key theorem 4.1, that if Y is affine, say $Y = \operatorname{Spec} B$, then we can compute the pushforwards of $\mathcal{F}$ by a complex of locally free modules

$$0 \to M^0 \to M^1 \to \cdots \to M^n \to 0$$

where in fact M^p is free for $p > 1$. Moreover, this computes pushforwards "universally": after a base change, this remains true.

Now the dimension of the left is uppersemicontinuous by uppersemicontinuity of fiber dimension of coherent sheaves. The semicontinuity theorem states that the dimension of the right is also uppersemicontinuous. More formally:

4.4. *Semicontinuity theorem. — Suppose* $X \to Y$ *is a projective morphisms of Noetherian schemes, and* $\mathcal{F}$ *is a coherent sheaf on* X *flat over* Y*. Then for each* $p \geq 0$*, the function* $Y \to \mathbb{Z}$ *given by* $y \mapsto \dim_{k(y)} H^p(X_y, \mathcal{F}_y)$ *is upper semicontinuous on* Y*.*

So "cohomology groups jump in projective flat families". Again, we can replace projective by proper once we've shown finite-dimensionality of higher pushforwards (which we haven't). For pedants: can the Noetherian hypotheses be excised?

Here is an example of jumping in action. Let C be a positive genus nonsingular projective irreducible curve, and consider the projection $\pi : E \times E \to E$. Let $\mathcal{L}$ be the invertible sheaf (line bundle) corresponding to the divisor that is the diagonal, minus the section $p_0 \in E$. then $\mathcal{L}_{p_0}$ is trivial, but $\mathcal{L}_p$ is non-trivial for any $p \neq p_0$ (as we've shown earlier in the "fun with curves" section). Thus $h^0(E, \mathcal{L}_p)$ is 0 in general, but jumps to 1 for $p = p_0$.

Remark. Deligne showed that in the smooth case, at least over $\mathbb{C}$, there is no jumping of cohomology of the structure sheaf.

Proof. The result is local on Y, so we may assume Y is affine. Let K^* be a complex as in the key theorem 4.1. By localizing further, we can assume K^* is locally free. *So we are computing cohomology on any fiber using a complex of vector bundles.*

Then for $y \in Y$

$$\begin{aligned} \dim_{k(y)} H^p(X_y, \mathcal{F}_y) &= \dim_{k(y)} \ker(d^p \otimes_A k(y)) - \dim_{k(y)} \operatorname{im}(d^{p-1} \otimes_A k(y)) \\ &= \dim_{k(y)} (K^p \otimes k(y)) - \dim_{k(y)} \operatorname{im}(d^p \otimes_A k(y)) - \dim_{k(y)} \operatorname{im}(d^{p-1} \otimes_A k(y)) \end{aligned}$$

(Side point: by taking alternating sums of these terms, we get a second proof of Theorem 1.1 that $\chi(X_y, \mathcal{F}_y) = \sum(-1)^i h^i(X_y, \mathcal{F}_y)$ is a constant function of y. I mention this because if extended the fact that higher cohomology of coherents is coherent under proper pushforwards, we'd also have Theorem 1.1 in this case.)

Now $\dim_{k(y)} \operatorname{im}(d^p \otimes_A k(y))$ is a lower semicontinuous function on Y. Reason: the locus where the dimension is less than some number q is obtained by setting all $q \times q$ minors of the matrix $K^p \to K^{p+1}$ to 0. So we're done! □

5. Line bundles are trivial in a Zariski-closed locus, and glimpses of the relative Picard scheme

(This was discussed on Thursday May 4, but fits in well here.)

5.1. *Proposition. — Suppose $\mathcal{L}$ is an invertible sheaf on an integral projective scheme X such that both $\mathcal{L}$ and $\mathcal{L}^\vee$ have non-zero sections. Then $\mathcal{L}$ is the trivial sheaf.*

As usual, "projective" may be replaced by "proper". The only fact we need (which we haven't proved) is that the only global functions on proper schemes are constants. (We haven't proved that. It follows easily from the valuative criterion of properness — but we haven't proved that either!)

Proof. Suppose s and t are the non-zero sections of $\mathcal{L}$ and $\mathcal{L}^\vee$. Then they are both non-zero at the generic point (or more precisely, in the stalk at the generic point). (Otherwise, they would be the zero-section — this is where we are using the integrality of X.) Under the map $\mathcal{L} \otimes \mathcal{L}^\vee \to \mathcal{O}$, $s \otimes t$ maps to st, which is also non-zero. But the only global functions (global sections of $\mathcal{O}_X$) are the constants, so st is a non-zero constant. But then s

is nowhere 0 (or else st would be somewhere zero), so $\mathcal{L}$ has a nowhere vanishing section, and hence is trivial (isomorphic to $\mathcal{O}_X$). $\square$

Now suppose $X \to Y$ is a flat projective morphism with integral fibers. (It is a "flat family of geometrically integral schemes".) Suppose that $\mathcal{L}$ is an invertible sheaf. Then the locus of $y \in Y$ where $\mathcal{L}_y$ is trivial on X_y is a closed set. Reason: the locus where $h^0(X_y, \mathcal{L}_y) \geq 1$ is closed by the Semicontinuity Theorem 4.4, and the same holes for the locus where $h^0(X_y, \mathcal{L}_y^\vee) \geq 1$.

(Similarly, if $\mathcal{L}'$ and $\mathcal{L}''$ are two invertible sheaves on the family X, the locus of points y where $\mathcal{L}'_y \cong \mathcal{L}''$ is a closed subset: just apply the previous paragraph to $\mathcal{L} := \mathcal{L}' \otimes (\mathcal{L}'')^\vee$.)

In fact, we can jazz this up: for any $\mathcal{L}$, there is in a natural sense a closed *subscheme* where $\mathcal{L}$ is trivial. More precisely, we have the following theorem.

5.2. *Seesaw Theorem. — Suppose $\pi : X \to Y$ is a projective flat morphism to a Noetherian scheme, all of whose fibers are geometrically integral schemes, and $\mathcal{L}$ is an invertible sheaf on X. Then there is a unique closed subscheme $Y' \hookrightarrow Y$ such that for any fiber diagram*

$$\begin{tikzcd} X \times_Y Z \arrow[r, "g"] \arrow[d, "\rho"] & X \arrow[d, "\pi"] \\ Z \arrow[r, "f"] & Y \end{tikzcd}$$

such that $g^*\mathcal{L} \cong \rho^*\mathcal{M}$ for some invertible sheaf $\mathcal{M}$ on Z, then f factors (uniquely) through $Y' \to Y$.

I want to make three comments before possibly proving this.

- I have no idea why it is called the seesaw theorem.

- As a special case, there is a "largest closed subscheme" on which the invertible sheaf is the pullback of a trivial invertible sheaf.

- Also, this is precisely the statement that the functor is representable $Y' \to Y$, and that this morphism is a closed immersion.

I'm not going to use this, so I won't prove it. But a slightly stripped down version of this appears in Mumford (p. 89), and you should be able to edit his proof so that it works in this generality.

There is a lesson I want to take away from this: this gives evidence for existence of a very important moduli space: the Picard scheme. The Picard scheme $\operatorname{Pic} X/Y \to Y$ is a scheme over Y which represents the following functor: Given any $T \to Y$, we have the set of invertible sheaves on $X \times_Y T$, modulo those invertible sheaves pulled back from T. In other words, there is a natural bijection between diagrams of the form

$$\begin{tikzcd} \mathcal{L} \arrow[d] & \\ X \times_T Y \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & Y \end{tikzcd}$$

and diagrams of the form

$$\begin{tikzcd} & \mathrm{Pic}_{X/Y} \arrow[d] \\ T \arrow[ur] \arrow[r] & Y. \end{tikzcd}$$

It is a hard theorem (due to Grothendieck) that (at least if Y is reasonable, e.g. locally Noetherian — I haven't consulted the appropriate references) $\mathrm{Pic}\, X/Y \to Y$ exists, i.e. that this functor is representable. In fact $\mathrm{Pic}\, X/Y$ is of finite type.

We've seen special cases before when talking about curves: if C is a geometrically integral curve over a field k, of genus g, $\mathrm{Pic}\, C = \mathrm{Pic}\, C/k$ is a dimension g projective nonsingular variety.

Given its existence, it is easy to check that $\mathrm{Pic}_{X/Y}$ is a group scheme over Y, using our functorial definition of group schemes.

5.3. *Exercise.* Do this!

The group scheme has a zero-section $0 : Y \to \mathrm{Pic}_{X/Y}$. This turns out to be a closed immersion. The closed subscheme produced by the Seesaw theorem is precisely the pullback of the 0-section. I suspect that you can use the Seesaw theorem to show that the zero-section *is* a closed immersion.

5.4. *Exercise.* Show that the Picard scheme for $X \to Y$ (with our hypotheses: the morphism is flat and projective, and the fibers are geometrically integral) is separated over Y by showing that it satisfies the valuative criterion of separatedness.

Coming up soon: Grauert's Theorem and Cohomology and base change!

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