FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 39 AND 40

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CONTENTS

These are notes from both class 39 and class 40.

Today: the Euler exact sequence. Discussion of nonsingular varieties over algebraically closed fields: Bertini's theorem, the Riemann-Hurwitz formula, and the (co)normal exact sequence for nonsingular subvarieties of nonsingular varieties.

We have now established the general theory of differentials, and we are now going to apply it.

1. PROJECTIVE SPACE AND THE EULER EXACT SEQUENCE

We next examine the differentials of projective space $\mathbb{P}^n_k$, or more generally $\mathbb{P}^n_A$ where A is an arbitrary ring. As projective space is covered by affine open sets, on which the differentials form a rank n locally free sheaf, $\Omega_{\mathbb{P}^n_A/A}$ is also a rank n locally free sheaf.

1.1. *Important Theorem (the Euler exact sequence). — The sheaf of differentials $\Omega_{\mathbb{P}^n_A/A}$ satisfies the following exact sequence*

$$0 \to \Omega_{\mathbb{P}^n_A} \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n_A} \to 0.$$

This is handy, because you can get a hold of Ω in a concrete way. Here is an explicit example, to give you practice.

1.2. *Exercise.* Show that $H^1(\mathbb{P}^n_A, T_{\mathbb{P}^n_A}) = 0$. (This later turns out to be an important calculation for the following reason. If X is a nonsingular variety, $H^1(X, T_X)$ parametrizes deformations of the variety. Thus projective space can't deform, and is "rigid".)

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Let's prove the Euler exact sequence. I find this an amazing fact, and while I can prove it, I don't understand in my bones why this is true. Maybe someone can give me some enlightenment.

Proof. (What's really going on in this proof is that we consider those differentials on $\mathbb{A}^{n+1}_A \setminus \{0\}$ that are pullbacks of differentials on $\mathbb{P}^n_A$.)

I'll describe a map $\mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}$, and later identify the kernel with $\Omega_{X/Y}$. The map is given by

$$(s_0, s_1, \dots, s_n) \mapsto x_0 s_0 + x_1 s_1 + \cdots + x_n s_n.$$

Note that this is a degree 1 map.

Now I have to identify the kernel of this map with differentials, and I can do this on each open set (so long as I do it in a way that works simultaneously for each open set). So let's consider the open set U_0, where $x_0 \neq 0$, and we have coordinates $x_{j/0} = x_j/x_0$ $(1 \leq j \leq n)$. Given a differential

$$f_1(x_{1/0}, \dots, x_{n/0})\ dx_{1/0} + \cdots + f_n(x_{1/0}, \dots, x_{n/0})\ dx_{n/0}$$

we must produce $n+1$ sections of $\mathcal{O}(-1)$. As motivation, let me just look at the first term, and pretend that the projective coordinates are actual coordinates.

$$\begin{aligned} f_1\ dx_{1/0} &= f_1\ d(x_1/x_0) \\ &= f_1 \frac{x_0\ dx_1 - x_1\ dx_0}{x_0^2} \\ &= -\frac{x_1}{x_0^2} f_1\ dx_0 + \frac{f_1}{x_0}\ dx_1 \end{aligned}$$

Note that x_0 times the "coefficient of dx_0" plus x_1 times the "coefficient of dx_1" is 0, and also both coefficients are of homogeneous degree -1. Motivated by this, we take:

$$f_1\ dx_{1/0} + \cdots + f_n\ dx_{n/0} \mapsto \left(-\frac{x_1}{x_0^2} f_1 - \cdots - \frac{x_n}{x_0^2} f_n, \frac{f_1}{x_0}, \frac{f_2}{x_0}, \cdots, \frac{f_n}{x_0} \right) \tag{1}$$

Note that over U_0, this indeed gives an injection of $\Omega_{\mathbb{P}^n_A}$ to $\mathcal{O}(-1)^{\oplus(n+1)}$ that surjects onto the kernel of $\mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_X$ (if $(g_0, \dots, g_n)$ is in the kernel, take $f_i = x_0 g_i$ for $i > 0$).

Let's make sure this construction, applied to two different coordinate patches (say U_0 and U_1) gives the same answer. (This verification is best ignored on a first reading.) Note that

$$\begin{aligned} f_1\ dx_{1/0} + f_2\ dx_{2/0} + \cdots &= f_1\ d\frac{1}{x_{0/1}} + f_2\ d\frac{x_{2/1}}{x_{0/1}} + \cdots \\ &= -\frac{f_1}{x_{0/1}^2} dx_{0/1} + \frac{f_2}{x_{0/1}} dx_{2/1} - \frac{f_2 x_{2/1}}{x_{0/1}^2} dx_{0/1} + \cdots \\ &= -\frac{f_1 + f_2 x_{2/1} + \cdots}{x_{0/1}^2} dx_{0/1} + \frac{f_2 x_1}{x_0} dx_{2/1} + \cdots. \end{aligned}$$

Under this map, the $dx_{2/1}$ term goes to the second factor (where the factors are indexed 0 through n) in $\mathcal{O}(-1)^{\oplus(n+1)}$, and yields f_2/x_0 as desired (and similarly for $dx_{j/1}$ for $j > 2$).

Also, the $dx_{0/1}$ term goes to the "zero" factor, and yields

$$(\sum_{j=1}^{n} f_i(x_i/x_1)/(x_0/x_1)^2)/x_1 = f_i x_i/x_0^2$$

as desired. Finally, the "first" factor must be correct because the sum over i of x_i times the ith factor is 0. $\square$

Generalizations of the Euler exact sequence are quite useful. We won't use them later this year, so I'll state them without proof. Note that the argument applies without change if $\operatorname{Spec} A$ is replaced by an arbitrary base scheme. The Euler exact sequence further generalizes in a number of ways. As a first step, suppose V is a rank $n+1$ locally free sheaf (or vector bundle) on a scheme X. Then $\Omega_{\mathbb{P}V/X}$ sits in an Euler exact sequence:

$$0 \to \Omega_{\mathbb{P}V/X} \to \mathcal{O}(-1) \otimes V^\vee \to \mathcal{O}_X \to 0$$

If $\pi : \mathbb{P}V \to X$, the map $\mathcal{O}(-1) \otimes V^\vee \to \mathcal{O}_X$ is induced by $V^\vee \otimes \pi_* \mathcal{O}(1) \cong (V^\vee \otimes V) \otimes \mathcal{O}_X \to \mathcal{O}_X$, where $V^\vee \otimes V \to A$ is the trace map.

For another generalization, fix a base field, and let $G(m, n+1)$ be the space of vector spaces of dimension m in an $(n+1)$-dimensional vector space V. (This is called the *Grassmannian*. We have not shown that this is actually a variety in any natural way, but it is. The case $m = 1$ is $\mathbb{P}^n$.) Then over $G(m, n+1)$ we have a short exact sequence of locally free sheaves

$$0 \to S \to V \otimes \mathcal{O}_{G(m,n+1)} \to Q \to 0$$

where $V \otimes \mathcal{O}_{G(m,n+1)}$ is a trivial bundle, and S is the "universal subbundle" (such that over a point $[V' \subset V]$ of the Grassmannian $G(m, n+1)$, $S|_{[V' \subset V]}$ is V if you can see what that means). Then

$$\Omega_{G(m,n+1)/k} \cong \underline{\operatorname{Hom}}(Q, S). \tag{2}$$

1.3. *Exercise.* In the case of projective space, $m = 1$, $S = \mathcal{O}(-1)$. Verify (2) in this case.

This Grassmannian fact generalizes further to Grassmannian bundles.

2. Varieties over algebraically closed fields

We'll now discuss differentials in the case of interest to most people: varieties over algebraically closed fields. I'd like to begin with a couple of remarks.

2.1. *Remark: nonsingularity may be checked at closed points.* Recall from the first quarter a deep fact about regular local rings that we haven't proved: Any localization of a regular local ring at a prime is again regular local ring. (For a reference, see Matsumura's *Commutative Algebra*, p. 139.) I'm going to continue to use this without proof. It is possible I'll write up a proof later. But in any case, if this bothers you, you could re-define nonsingularity of locally finite type schemes over fields to be what other people call "nonsingularity at closed points", and the results of this section will hold.

2.2. *Remark for non-algebraically closed people.* Even if you are interested in non-algebraically closed fields, this section should still be of interest to you. In particular, if X is a variety over a field k, and $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, then $X_{\overline{k}}$ nonsingular implies that X is nonsingular. (You may wish to prove this yourself. By Remark 2.1, it suffices to check at closed points.) *Possible exercise.* In fact if k is separably closed, then $X_{\overline{k}}$ is nonsingular if and only if X is nonsingular, but this is a little bit harder.

Suppose for the rest of this section that X is a pure n-dimensional locally finite type scheme over an algebraically closed field k (e.g. a k-variety).

2.3. *Proposition. — $\Omega_{X/k}$ is locally free of rank n if and only if X is nonsingular.*

Proof. By Remark 2.1, it suffices to prove that $\Omega_{X/k}$ is locally free of rank n if and only if the closed points of X is nonsingular. Now $\Omega_{X/k}$ is locally free of rank n if and only if its fibers at all the closed points are rank n (recall that fibers jump in closed subsets). As the fiber of the cotangent sheaf is canonically isomorphic to the Zariski tangent space at closed points (done earlier), the Zariski tangent space at every closed point must have dimension n, i.e. the closed points are all nonsingular. □

Using this Proposition, we can get a new result using a neat trick.

2.4. *Theorem. — If X is integral, there is an dense open subset U of X which is nonsingular.*

Proof. The $n = 0$ case is immediate, so we assume $n > 0$.

We will show that the rank at the generic point is n. Then by uppersemicontinuity of the rank of a coherent sheaf (done earlier), it must be n in an open neighborhood of the generic point, and we are done by Proposition 2.3.

We thus have to check that if K is the fraction field of a dimension n integral finite-type k-scheme, i.e. if K is a transcendence degree n extension of k, then $\Omega_{K/k}$ is an n-dimensional vector space. But any transcendence degree $n > 1$ extension is separably generated: we can find n algebraically independent elements of K over k, say $x_1, \dots, x_n$, such that $K/k(x_1, \dots, x_n)$ is separable. (This is a fact about transcendence theory.) Then $\Omega_{K/k}$ is generated by $dx_1, \dots, dx_n$ (as $dx_1, \dots, dx_n$ generate $\Omega_{k(x_1,\dots,x_n)/k}$, and any element of K is separable over $k(x_1, \dots, x_n)$ — this is summarized most compactly using the affine form of the relative cotangent sequence). □

2.5. *Bertini's Theorem. — Suppose X is a nonsingular closed subvariety of $\mathbb{P}^n_k$ (where the standing hypothesis for this section, that k is algebraically closed, holds). Then there is an open subset of hyperplanes H of $\mathbb{P}^n_k$ such that H doesn't contain any component of X, and the scheme $H \cap X$ is a nonsingular variety. More precisely, this is an open subset of the dual projective space $\mathbb{P}^{n\vee}_k$. In particular, there exists a hyperplane H in $\mathbb{P}^n_k$ not containing any component of X such that the scheme $H \cap X$ is also a nonsingular variety.*

(We've already shown in our section on cohomology that if X is connected, then $H \cap X$ is connected.)

We may have used this before to show the existence of nonsingular curves of any genus, for example, although I don't think we did. (We discussed Bertini in class 35, p. 4.)

Note that this implies that a general degree $d > 0$ hypersurface in $\mathbb{P}^n_k$ also intersects X in a nonsingular subvariety of codimension 1 in X: replace $X \hookrightarrow \mathbb{P}^n$ with the composition $X \hookrightarrow \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ where the latter morphism is the dth Veronese map.

Proof. In order to keep the language of the proof as clean as possible, I'll assume X is irreducible, but essentially the same proof applies in general.

The central idea of the proof is quite naive and straightforward. We'll describe the hyperplanes that are "bad", and show that they form a closed subset of dimension at most $n-1$ of $\mathbb{P}^{n\vee}_k$, and hence that the complement is a dense open subset. More precisely, we will define a projective variety $Y \subset X \times \mathbb{P}^{n\vee}_k$ that will be:

$$Y = \{(p \in X, H \subset \mathbb{P}^n_k) : p \in H, p \text{ is a singular point of } H \cap X, \text{ or } X \subset H\}$$

We will see that $\dim Y \leq n-1$. Thus the image of Y in $\mathbb{P}^{n\vee}_k$ will be a closed subset (the image of a closed subset by a projective hence closed morphism!), of dimension of $n-1$, and its complement is open.

We'll show that Y has dimension $n-1$ as follows. Consider the map $Y \to X$, sending (p, H) to p. Then a little thought will convince you that there is a $(n - \dim X - 1)$-dimensional family of hyperplanes through $p \in X$ such that $X \cap H$ is singular at p, or such that X is contained in H. (Those two conditions can be summarized quickly as: H contains the "first-order formal neighborhood of p in X", $\operatorname{Spec} \mathcal{O}_{X,p}/\mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$.) Hence we expect Y to be a projective bundle, whose fibers are dimension $n - \dim X - 1$, and hence that Y has dimension at most $\dim X + (n - \dim X - 1) = n - 1$. In fact this is the case, but we'll show a little less (e.g. we won't show that $Y \to X$ is a projective bundle) because we don't need to prove this full statement to complete our proof of Bertini's theorem.

Let's put this strategy into action. We first define Y more precisely, in terms of equations on $\mathbb{P}^n \times \mathbb{P}^{n\vee}$, where the coordinates on $\mathbb{P}^n$ are $x_0, \dots x_n$, and the dual coordinates on $\mathbb{P}^{n\vee}$ are $a_0, \dots, a_n$. Suppose X is cut out by $f_1, \dots, f_r$. (We will soon verify that this definition of Y is independent of these equations.) Then we take these equations as some of the defining equations of Y. (So far we have defined the subscheme $X \times \mathbb{P}^{n\vee}$.) We also add the equation $a_0 x_0 + \cdots + a_n x_n = 0$. (So far we have described the subscheme of $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ corresponding to points (p, H) where $p \in X$ and $p \in H$.) Note that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix}$$

has corank equal to $\dim X$ at all closed points of X — this is precisely the Jacobian condition for nonsingularity (class 12, p. 3, 1.6). (Although we won't use this fact, in fact it has that corank $\dim X$ everywhere on X. Reason: the locus where the corank jumps is a closed locus, as this is described by equations, namely determinants of minors. Thus as the corank is constant at all closed points, it is constant everywhere.) We then require that the Jacobian matrix with a new row $(a_0, \cdots, a_n)$ has corank $\geq \dim X$ (hence $= \dim X$). This is cut out by equations (determinants of minors). By the Jacobian description of the Zariski tangent space, this condition encodes the requirement that the Zariski tangent space of $H \cap X$ at p has dimension precisely $\dim X$, which is $\dim H \cap X + 1$ (i.e. $H \cap X$ is singular at p) if H does not contain X, or if H contains X. This is precisely the notion that we hoped to capture.

Before getting on with our proof, let's do an example to convince ourselves that this algebra is describing the geometry we desire. Consider the plane conic $x_0^2 - x_1^2 - x_2^2 = 0$ over a field of characteristic not 2, which I picture as the circle $x^2 + y^2 = 1$ from the real picture in the chart U_0. (At this point I drew a picture.) Consider the point $(1, 1, 0)$, corresponding to $(1, 0)$ on the circle. We expect the tangent line in the affine plane to be $x = 1$, which should correspond to $x_0 - x_1 = 0$. Let's see what the algebra gives us. The Jacobian matrix is $\begin{pmatrix} 2x_0 & -2x_1 & -2x_2 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 0 \end{pmatrix}$, which indeed has rank 1 as expected. Our recipe asks that the matrix $\begin{pmatrix} 2 & -2 & 0 \\ a_0 & a_1 & a_2 \end{pmatrix}$ have rank 1, which means that $(a_0, a_1, a_2) = (a_0, -a_0, 0)$, and also that $a_0 x_0 + a_1 x_1 + a_2 x_2 = 0$, which is precisely what we wanted!

Returning to our construction, we can see that the Y just described is independent of the choice of $f_1, \ldots, f_r$ (although we won't need this fact).

Here's why. It suffices to show that if we add in a redundant equation (some homogeneous f_0 that is a $k[x_0, \ldots, x_n]$-linear combination of the f_i), we get the same Y (as then if we had a completely different set of f's, we could add them in one at a time, and then remove the old f's one at a time). If we add in a redundant equation, then that row in the Jacobian matrix will be a $k[x_0, \ldots, x_n]$-linear combination of other rows, and thus the rank remains unchanged. (There is a slight issue I am glossing over here — f_0 may vanish on Y despite not being a linear combination of $f_1, \ldots, f_n$.)

We'll next show that $\dim Y = n - 1$. For each $p \in X$, let Z_p be the locus of hyperplanes containing p, such that $H \cap X$ is singular at p, or else contains all of X; what is the dimension of Z_p? (For those who have heard of these words: what is the dimension of the locus of hyperplanes containing a first-order formal neighborhood of p in X?) Suppose $\dim X = d$. Then this should impose $d + 1$ conditions on hyperplanes. This means that it is a codimension $d + 1$, or dimension $n - d - 1$, projective space. Thus we should expect $Y \to X$ to be a projective bundle of relative dimension $n - d - 1$ over a variety of dimension d, and hence that $\dim Y = n - 1$. For convenience, I'll verify a little less: that $\dim Y \leq n - 1$.

Suppose Y has dimension N. Let $H_1, \ldots, H_d$ be general hyperplanes such that $H_1 \cap \cdots \cap H_d \cap X$ is a finite set of points (this was an exercise from long ago, class 31, ex. 1.5, p. 4). Then if $\pi : Y \to X$ is the projection to X, then (using Krull's Principal Ideal Theorem)

$$n - d - 1 = \dim Y \cap \pi^* H_1 \cap \cdots \cap \pi^* H_d \geq \dim Y - d$$

from which $\dim Y \leq n - 1$. $\square$

2.6. *Exercise-.* Show that Bertini's theorem still holds even if X is singular in dimension 0. (This isn't that important.)

2.7. *Remark..* The image in $\mathbb{P}^n$ tends to be a divisor. This is classically called the *dual variety*. The following exercise will give you some sense of it.

2.8. *Exercise.* Suppose $C \subset \mathbb{P}^2$ is a nonsingular conic over a field of characteristic not 2. Show that the dual variety is also a conic. (More precisely, suppose C is cut out by $f(x_0, x_1, x_2) = 0$. Show that $\{(a_0, a_1, a_2) : a_0 x_0 + a_1 x_1 + a_2 x_2 = 0\}$ is cut out by a quadratic equation.) Thus for example, through a general point in the plane, there are two tangents to C. (The points on a line in the dual plane corresponds to those lines through a point of the original plane.)

We'll soon find the degree of the dual to a degree d curve (after we discuss the Riemann-Hurwitz formula), at least modulo some assumptions.

2.9. The Riemann-Hurwitz formula.

We're now ready to discuss and prove the Riemann-Hurwitz formula. We continue to work over an algebraically closed field k. Everything below can be mildly modified to work for a perfect field, e.g. any field of characteristic 0, and I'll describe this at the end of the discussion (Remark 2.17).

Definition (separable morphisms). A finite morphism between integral schemes $f : X \to Y$ is said to be *separable* if it is dominant, and the induced extension of function fields $\mathrm{FF}(X)/\mathrm{FF}(Y)$ is a separable extension. (Similarly, a generically finite morphism is *generically separable* if it is dominant, and the induced extension of function fields is a separable extension. We may not use this notion.) Note that this comes for free in characteristic 0.

2.10. *Proposition. — If $f : X \to Y$ is a finite separable morphism of nonsingular integral curves, then we have an exact sequence*

$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0.$$

Proof. We have right-exactness by the relative cotangent sequence, so we need to check only that $\phi : f^*\Omega_{Y/k} \to \Omega_{X/k}$ is injective. Now $\Omega_{Y/k}$ is an invertible sheaf on Y, so $f^*\Omega_{Y/k}$ is an invertible sheaf on X. Thus it has no torsion subsheaf, so we need only check that ϕ is an inclusion at the generic point. We thus tensor with $\mathcal{O}_\eta$ where η is the generic point of X. This is an exact functor (it is localization), and $\mathcal{O}_\eta \otimes \Omega_{X/Y} = 0$ (as $\mathrm{FF}(X)/\mathrm{FF}(Y)$ is a separable by hypothesis, and Ω for separable field extensions is 0 by Ex. 2.10, class 37, which was also Ex. 4, problem set 17). Also, $\mathcal{O}_\eta \otimes f^*\Omega_{Y/k}$ and $\mathcal{O}_\eta \otimes \Omega_{X/k}$ are both one-dimensional $\mathcal{O}_\eta$-vector spaces (they are the stalks of invertible sheaves at the generic point). Thus by considering

$$\mathcal{O}_\eta \otimes f^*\Omega_{Y/k} \to \mathcal{O}_\eta \otimes \Omega_{X/k} \to \mathcal{O}_\eta \otimes \Omega_{X/Y} \to 0$$

(which is
$$\mathcal{O}_\eta \to \mathcal{O}_\eta \to 0 \to 0)$$
we see that $\mathcal{O}_\eta \otimes f^*\Omega_{Y/k} \to \mathcal{O}_\eta \otimes \Omega_{X/k}$ is injective, and thus that $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is injective. $\square$

2.11. It is worth noting what goes wrong for non-separable morphisms. For example, suppose k is a field of characteristic p, consider the map $f : \mathbb{A}^1_k = \operatorname{Spec} k[t] \to \mathbb{A}^1_k = \operatorname{Spec} k[u]$ given by $u = t^p$. Then $\Omega_{\mathbb{A}^1_k/k}$ is the trivial invertible sheaf generated by dt. As another (similar but different) example, if $K = k(x)$ and $K' = K(x^p)$, then the inclusion $K' \hookrightarrow K$ induces $f : \operatorname{Spec} K[t] \to \operatorname{Spec} K'[t]$. Once again, Ω_f is an invertible sheaf, generated by dx (which in this case is pulled back from $\Omega_{K/K'}$ on $\operatorname{Spec} K$). In both of these cases, we have maps from one affine line to another, and there are vertical tangent vectors.

2.12. The sheaf $\Omega_{X/Y}$ on the right side of Proposition 2.10 is a coherent sheaf not supported at the generic point. Hence it is supported at a finite number of points. These are called the *ramification points* (and the images downstairs are called the *branch points*). I drew a picture here.

Let's check out what happens at closed points. We have two discrete valuation rings, say $\operatorname{Spec} A \to \operatorname{Spec} B$. I've assumed that we are working over an algebraically closed field k, so this morphism $B \to A$ induces an isomorphism of residue fields (with k). Suppose their uniformizers are s and t respectively, with $t \mapsto us^n$ where u is a unit of A. Then
$$dt = d(us^n) = uns^{n-1}\, ds + s^n\, du.$$
This vanishes to order at least $n-1$, and precisely $n-1$ if n doesn't divide the characteristic. The former case is called *tame* ramification, and the latter is called *wild* ramification. We call this order the *ramification order* at this point of X.

Define the *ramification divisor* on X as the sum of all points with their corresponding ramifications (only finitely many of which are non-zero). The image of this divisor on Y is called the *branch divisor*.

2.13. *Straightforward exercise: interpreting the ramification divisor in terms of number of preimages.* Suppose all the ramification above $y \in Y$ is tame. Show that the degree of the branch divisor at y is $\deg(f : X \to Y) - \#f^{-1}(y)$. Thus the multiplicity of the branch divisor counts the extent to which the number of preimages is less than the degree.

2.14. *Proposition. — Suppose R is the ramification divisor of $f : X \to Y$. Then $\Omega_X(-R) \cong f^*\Omega_Y$.*

Note that we are making no assumption that X or Y is projective.

Proof. This says that we can interpret the invertible sheaf $f^*\Omega_Y$ over an open set of X as those differentials on X vanishing along the ramification divisor. But that is the content of Proposition 2.10. $\square$

Then the Riemann-Hurwitz formula follows!

2.15. *Theorem (Riemann-Hurwitz).* — *Suppose* $f : X \to Y$ *is a finite separable morphism of curves. Let* $n = \deg f$. *Then* $2g(X) - 2 = n(2g(Y) - 2) + \deg R$.

Note that we now need the projective hypotheses in order to take degrees of invertible sheaves.

Proof. This follows by taking the degree of both sides of Proposition 2.14 (and using the fact that the pullback of a degree d line bundle by a finite degree n morphism is dn, which was an earlier exercise, Ex. 3.1, class 29, p. 3, or Ex. 2, problem set 13). □

2.16. *Exercise: degree of dual curves.* Describe the degree of the dual to a nonsingular degree d plane curve C as follows. Pick a general point $p \in \mathbb{P}^2$. Find the number of tangents to C through p, by noting that projection from p gives a degree d map to $\mathbb{P}^1$ (why?) by a curve of known genus (you've calculated this before), and that ramification of this cover of $\mathbb{P}^1$ corresponds to a tangents through p. (Feel free to make assumptions, e.g. that for a general p this branched cover has the simplest possible branching — this should be a back-of-an-envelope calculation.)

2.17. *Remark: Riemann-Hurwitz over perfect fields.* This discussion can be extended to work when the base field is not algebraically closed; perfect will suffice. The place we assumed that the base field was algebraically closed was after we reduced to understanding the ramification of the morphism of the spectrum of one discrete valuation ring over our base field k to the spectrum of another, and we assumed that this map induced an isomorphism of residue fields. In general, it can be a finite extension. Let's analyze this case explicitly. Consider a map $\operatorname{Spec} A \to \operatorname{Spec} B$ of spectra of discrete valuation rings, corresponding to a ring extension $B \to A$. Let s be the uniformizer of A, and t the uniformizer of B. Let $\mathfrak{m}$ be the maximal ideal of A, and $\mathfrak{n}$ the maximal ideal of A. Then as $A/\mathfrak{m}$ is a finite extension of $B/\mathfrak{n}$, it is generated over $B/\mathfrak{n}$ by a single element (we're invoking here the theorem of the primitive element, and we use the "perfect" assumption here). Let s' be any lift of this element of $A/\mathfrak{m}$ to A. Then A is generated over B by s and s', so $\Omega_{A/B}$ is generated by ds and ds'. The contribution of ds is as described above. You can show that $ds' = 0$. Thus all calculations above carry without change, except for the following.

(i) We have to compute the degree of the ramification divisor appropriately: we need to include as a factor the degree of the field extension of the residue field of the point on the *source* (over k).

(ii) Exercise 2.13 doesn't work, but can be patched by replacing $\#f^{-1}(y)$ with the number of *geometric* preimages.

As an example of what happens differently in (ii), consider the degree 2 finite morphism $X = \operatorname{Spec} \mathbb{Z}[i] \to Y = \operatorname{Spec} \mathbb{Z}$. We can compute $\Omega_{\mathbb{Z}[i]/\mathbb{Z}}$ directly, as $\mathbb{Z}[i] \cong \mathbb{Z}[x]/(x^2+1)$: $\Omega_{\mathbb{Z}[i]/\mathbb{Z}} \cong \mathbb{Z}[i]dx/(2dx)$. In other words, it is supported at the prime $(1+i)$ (the unique prime above $[(2)] \in \operatorname{Spec} \mathbb{Z}$). However, the number of preimages of points in $\operatorname{Spec} \mathbb{Z}$ is not always 2 away from the point $[(2)]$; half the time (including, for example, over $[(3)]$) there is one point, but the field extension is separable.

2.18. *Exercise (aside): Artin-Schreier covers.* In characteristic 0, the only connected unbranched cover of $\mathbb{A}^1$ is the isomorphism $\mathbb{A}^1 \xrightarrow{\sim} \mathbb{A}^1$; that was an earlier example/exercise, when we discussed Riemann-Hurwitz the first time. In positive characteristic, this needn't be true, because of wild ramification. Show that the morphism corresponding to $k[x] \to k[x,y]/(y^p - x^p - y)$ is such a map. (Once the theory of the algebraic fundamental group is developed, this translates to: "$\mathbb{A}^1$ is not simply connected in characteristic p.")

2.19. The conormal exact sequence for nonsingular varieties.

Recall the conormal exact sequence. Suppose $f : X \to Y$ morphism of schemes, $Z \hookrightarrow X$ closed subscheme of X, with ideal sheaf $\mathcal{I}$. Then there is an exact sequence of sheaves on Z:

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0.$$

I promised you that in good situations this is exact on the left as well, as our geometric intuition predicts. Now let $Z = \operatorname{Spec} k$ (where $k = \overline{k}$), and Y a nonsingular k-variety, and $X \subset Y$ an irreducible closed subscheme cut out by the quasicoherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$.

2.20. *Theorem (conormal exact sequence for nonsingular varieties). — X is nonsingular if and only if (i) $\Omega_{X/k}$ is locally free, and (ii) the conormal exact sequence is exact on the left also:*

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0.$$

Moreover, if Y is nonsingular, then $\mathcal{I}$ is locally generated by $\operatorname{codim}(X,Y)$ elements, and $\mathcal{I}/\mathcal{I}^2$ is a locally free of rank $\operatorname{codim}(X,Y)$.

This latter condition is the definition of something being a *local complete intersection* in a nonsingular scheme.

You can read a proof of this in Hartshorne II.8.17. I'm not going to present it in class, as we'll never use it. The only case I've ever seen used is the implication that if X is nonsingular, then (i) and (ii) hold; and we've already checked (i). This implication (that in the case of a nonsingular subvariety of a nonsingular variety, the conormal and hence normal exact sequence is exact) is very useful for relating the differentials on a nonsingular subvariety to the normal bundle.

The real content is that in the case of a nonsingular subvariety of a nonsingular variety, the conormal exact sequence is exact on the left as well, and in this nice case we have a short exact sequence of locally free sheaves (vector bundles). By dualizing, i.e. applying $\underline{\operatorname{Hom}}(\cdot, \mathcal{O}_X)$, we obtain the *normal exact sequence*

$$0 \to T_{X/k} \to T_{Y/k} \to \mathcal{N}_{X/Y} \to 0$$

which is very handy. Note that dualizing an exact sequence will give you a left-exact sequence in general, but dualizing an exact sequence of locally free sheaves will always be locally free. (In fact, all you need is that the third term is locally free. I could make this an exercise; it may also follow if I define Ext soon after defining Tor, as an exercise.)

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