FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 34 CRIB SHEET

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This is a summary of useful facts we proved or assumed. We will use them in the next two classes.

All curves C are projective, and geometrically integral and nonsingular over a field k.

There is an invertible sheaf (rank bundle) $\mathcal{K}$, called the *dualizing sheaf*; it is also the sheaf of differentials (in this guise it is called $\Omega_{C/k}$), and the cotangent bundle. $\deg \mathcal{K} = 2g - 2$.

The *Riemann-Hurwitz formula* is $2g_C - 2 = d(2g_{C'} - 2) + \deg R$, where R is the *ramification divisor*.

Serre duality. There is an isomorphism $H^0(C, \mathcal{K}) \xrightarrow{\sim} k$ For any coherent sheaf $\mathcal{F}$, the natural map

$$\boxed{H^0(C, \mathcal{F}) \otimes_k H^1(C, \mathcal{K} \otimes \mathcal{F}^\vee) \to H^0(C, \mathcal{K})}$$

is a perfect pairing, so in particular, $h^0(C, \mathcal{F}) = h^1(C, \mathcal{K} \otimes \mathcal{F}^\vee)$. (As $g := h^1(C, \mathcal{O}_C)$, we get $h^0(C, \mathcal{K}) = g$ as well.) Hence Riemann-Roch now states:

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg \mathcal{L} - g + 1.$$

Applying this to $\mathcal{L} = \mathcal{K}$, we get $\deg \mathcal{K} = 2g - 2$ (promised earlier).

Suppose now that $\mathcal{L}$ is an invertible sheaf on C.

0.1. $h^0(C, \mathcal{L}) = 0$ if $\deg \mathcal{L} < 0$. $h^0(C, \mathcal{L}) = 0$ or 1 if $\deg \mathcal{L} = 0$.

0.2. Suppose p is any closed point of degree 1. (In other words, the residue field of p is k.) Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 0$ or 1.

0.3. Suppose for this remark that k is algebraically closed. (In particular, *all* closed points have degree 1 over k.) Then if $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 1$ for *all* closed points p, then $\mathcal{L}$ is base-point-free, and hence induces a morphism from C to projective space.

0.4. Suppose p and q are distinct points of degree 1. Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 0$, 1, or 2. If $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$, then $\mathcal{L}$ separates points p and q, by which I mean that the corresponding map f to projective space satisfies $f(p) \neq f(q)$.

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0.5. If p is a point of degree 1, then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-2p)) = 0$, 1, or 2. If it is 2, then the map corresponding to $\mathcal{L}$ separates the tangent vectors at p.

0.6. Combining some of our previous comments: suppose C is a curve over an *algebraically closed* field k, and $\mathcal{L}$ is an invertible sheaf such that for *all* closed points p and q, *not necessarily distinct*, $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$, then $\mathcal{L}$ gives a closed immersion into projective space.

0.7. We now bring in Serre duality. $\deg \mathcal{L} > 2g - 2$ implies

$$\boxed{h^0(C, \mathcal{L}) = \deg \mathcal{L} - g - 1.}$$

If $\mathcal{L}$ is a degree $2g-2$ invertible sheaf, then $\mathcal{L}$ has $g-1$ or g sections, and it has g sections if and only if $\mathcal{L} \cong \mathcal{K}$.

0.8. *Our most important conclusion.* $\deg \mathcal{L} \geq 2g$ implies that $\mathcal{L}$ is basepoint free (and hence determines a morphism to projective space). Also, $\deg \mathcal{L} \geq 2g+1$ implies that this is in fact a closed immersion. Remember this!

0.9. Suppose C is not isomorphic to $\mathbb{P}^1_k$ (with no restrictions on the genus of C), and $\mathcal{L}$ is an invertible sheaf of degree 1. Then $h^0(C, \mathcal{L}) < 2$.

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