FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 33

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CONTENTS

Last day: Applications of higher pushforwards; crash course in spectral sequences.

Today: The Leray spectral sequence. Beginning fun with curves: the Riemann-Hurwitz formula.

Before I start, here is one small comment I should have made earlier. In the notation $R^j f_* \mathcal{F}$ for higher pushforward sheaves, the "R" stands for "right derived functor", and "corresponds" to the fact that we get a long exact sequence in cohomology extending to the right (from the 0th terms). More generally, next quarter we will see that in good circumstances, if we have a left-exact functor, there may be a long exact sequence going off to the right, in terms of right derived functors. Similarly, if we have a right-exact functor (e.g. if M is an A-module, then $\otimes_A M$ is a right-exact functor from the category of A-modules to itself), there may be a long exact sequence going off to the left, in terms of left derived functors.

Here is another exercise that I should have asked earlier. I have also now included it in the class 32 notes (in section 1).

Exercise. Suppose that X is a quasicompact separated k-scheme, where k is a field. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, and $f : X_{\overline{k}} \to X$ the projection. Describe a natural isomorphism $H^i(X, \mathcal{F}) \otimes_k \overline{k} \to H^i(X_{\overline{k}}, f^* \mathcal{F})$. Recall that a k-scheme X is *geometrically integral* if $X_{\overline{k}}$ is integral. Show that if X is geometrically integral and projective, then $H^0(X, \mathcal{O}_X) \cong k$. (This is a clue that $\mathbb{P}^1_{\mathbb{C}}$ is not a geometrically integral $\mathbb{R}$-scheme.)

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1. Leray spectral sequence

Suppose

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & Z, & \end{array}$$

with f and g (and hence h) quasicompact and separated. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. The Leray spectral sequence lets us find out about the higher pushforwards of h in terms of the higher pushforwards under g of the higher pushforwards under f.

1.1. *Theorem (Leray spectral sequence). — There is a spectral sequence whose $E_2^{p,q}$-term is $R^j g_*(R^i f_* \mathcal{F})$, abutting to $R^{i+j} h_* \mathcal{F}$.*

An important special case is if $Z = \operatorname{Spec} k$, or Z is some other base ring. Then this gives us handle on the cohomology of $\mathcal{F}$ on X in terms of the cohomology of its higher pushforwards to Y.

Proof. We assume Z is an affine ring, say $\operatorname{Spec} A$. Our construction will be "natural" and will hence glue. (At worst, we you can check that it behaves well under localization.)

Fix a finite affine cover of X, U_i. Fix a finite affine cover of Y, V_j. Create a double complex

$$E_0^{a,b} = \oplus_{|I|=a+1,|J|=b+1} \mathcal{F}(U_I \cap \pi^{-1} V_J)$$

for $a, b \geq 0$, with obvious Cech differential maps. By exercise 15 on problem set 11 (class 25, exercise 1.31), $U_I \cap \pi^{-1} V_J$ is affine (for all I, J).

Let's choose the filtration that corresponds to first taking the arrow in the vertical (V) direction. For each I, we'll get a Cech covering of U_I. The Cech cohomology of an affine is trivial except for H^0, so the E_1 term will be 0 except when $j = 0$. There, we'll get $\oplus \mathcal{F}(U_I)$. Then the E_2 term will be $E_2^{p,q} = H^p(X, \mathcal{F}) = \Gamma(Z, R^p h_* \mathcal{F})$ if $q = 0$ and 0 otherwise, and it will converge there.

Let's next choose the filtration that corresponds to first taking the arrow in the horizontal (U) direction. For each V_J, we will get a Cech covering of $\pi^{-1} V_J$. The entries of E_1 will thus be $\oplus_J H^i(f^{-1} V_j, \mathcal{F}) = \oplus_j \Gamma(V_j, R^i \pi_* \mathcal{F})$. Thus E_2 will be as advertised in the statement of Leray. $\square$

Here are some useful examples.

Consider $h^i(\mathbb{P}_k^m \times_k \mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^m \times_k \mathbb{P}_k^n})$. We get 0 unless $i = 0$, in which case we get 1. (The same argument shows that $h^i(\mathbb{P}_A^m \times_A \mathbb{P}_A^n, \mathcal{O}_{\mathbb{P}_A^m \times_A \mathbb{P}_A^n}) \cong A$ if $i = 0$, and 0 otherwise.) You should make this precise:

Exercise. Suppose Y is any scheme, and $\pi : \mathbb{P}^n_Y \to Y$ is the trivial projective bundle over Y. Show that $\pi_* \mathcal{O}_{\mathbb{P}^n_Y} \cong \mathcal{O}_Y$. More generally, show that $R^j \pi_* \mathcal{O}(m)$ is a finite rank free sheaf on Y, and is 0 if $j \neq 0, n$. Find the rank otherwise.

More generally, let's consider $H^i(\mathbb{P}^m_k \times_k \mathbb{P}^n_k, \mathcal{O}(a,b))$. I claim that for each (a,b) at most one cohomology group is non-trivial, and it will be $i = 0$ if $a, b \geq 0$; $i = m+n$ if $a \leq -m-1$, $b \leq -n-1$; $i = m$ if $a \geq 0$, $b \leq -n-1$, and $i = n$ if $a \leq -m-1$, $b = 0$. I attempted to show this to you in a special case, in the hope that you would see how the argument goes. I tried to show that $h^i(\mathbb{P}^2_k \times_k \mathbb{P}^1_k, \mathcal{O}(-4,1))$ is 6 if $i = 2$ and 0 otherwise. The following exercise will help you see if you understood this.

Exercise. Let A be any ring. Suppose a is a negative integer and b is a positive integer. Show that $H^i(\mathbb{P}^m_A \times_A \mathbb{P}^n_A, \mathcal{O}(a,b))$ is 0 unless $i = m$, in which case it is a free A-module. Find the rank of this free A-module. (Hint: Use the previous exercise, and the projection formula, which was Exercise 1.3 of class 32, and exercise 17 of problem set 14.)

We can now find curves of any (non-negative) genus, over any algebraically closed field. An integral projective nonsingular curve over k is *hyperelliptic* if admits a finite degree 2 morphism (or "cover") of $\mathbb{P}^1$.

1.2. *Exercise.* (a) Find the genus of a curve in class $(2,n)$ on $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$. (A curve in class $(2,n)$ is any effective Cartier divisor corresponding to invertible sheaf $\mathcal{O}(2,n)$. Equivalently, it is a curve whose ideal sheaf is isomorphic to $\mathcal{O}(-2,-n)$. Equivalently, it is a curve cut out by a non-zero form of bidegree $(2,n)$.)
(b) Suppose for convenience that k is algebraically closed of characteristic not 2. Show that there exists an integral nonsingular curve in class $(2,n)$ on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ for each $n > 0$.

1.3. *Exercise.* Suppose X and Y are projective k-schemes, and $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on X and Y respectively. Recall that if $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the two projections, then $\mathcal{F} \boxtimes \mathcal{G} := \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{G}$. Prove the following, adding additional hypotheses if you find them necessary.
(a) Show that $H^0(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^0(X, \mathcal{F}) \otimes H^0(Y, \mathcal{G})$.
(b) Show that $H^{\dim X + \dim Y}(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^{\dim X}(X, \mathcal{F}) \otimes_k H^{\dim Y}(Y, \mathcal{G})$.
(c) Show that $\chi(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \chi(X, \mathcal{F}) \chi(Y, \mathcal{G})$.

I suspect that this Leray spectral sequence converges in this case at E^2, so that $h^n(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \sum_{i+j=n} h^i(X, \mathcal{F}) h^j(Y, \mathcal{G})$. Or if this is false, I'd like to see a counterexample. It might even be true that

$$H^n(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \oplus_{i+j=n} H^i(X, \mathcal{F}) \otimes H^j(Y, \mathcal{G}).$$

2. Fun with Curves

We already know enough to study curves in a great deal of detail, so this seems like a good way to end this quarter. We get much more mileage if we have a few facts involving differentials, so I'll introduce these facts, and take them as a black box. The actual black boxes we'll need are quite small, but I want to tell you some of the background behind them.

For this topic, we will assume that all curves are projective geometrically integral non-singular curves over a field k. We will sometimes add the hypothesis that k is algebraically closed.

Most people are happy with working over algebraically closed fields, and all of you should ignore the adverb "geometrically" in the previous paragraph. For those interested in non-algebraically closed fields, an example of a curve that is integral but not geometrically integral is $\mathbb{P}^1_{\mathbb{C}}$ over $\mathbb{R}$. Upon base change to the algebraic closure $\mathbb{C}$ of $\mathbb{R}$, this curve has two components.

2.1. **Differentials on curves.** There is a sheaf of differentials on a curve C, denoted Ω_C, which is an invertible sheaf. (In general, nonsingular k-varieties of dimension d will have a sheaf of differentials over k that will be locally free of rank k. And differentials will be defined in vastly more generality.) We will soon see that this invertible sheaf has degree equal to twice the genus minus 2: $\boxed{\deg \Omega_C = 2g_c - 2}$. For example, if $C = \mathbb{P}^1$, then $\Omega_C \cong \mathcal{O}(-2)$.

Differentials pull back: any surjective morphism of curves $f : C \to C'$ induces a natural map $f^*\Omega_{C'} \to \Omega_C$.

2.2. **The Riemann-Hurwitz formula.** Whenever we invoke this formula (in this section), we will assume that k is algebraically closed and characteristic 0. These conditions aren't necessary, but save us some extra hypotheses. Suppose $f : C \to C'$ is a dominant morphism. Then it turns out $f^*\Omega_{C'} \hookrightarrow \Omega_C$ is an inclusion of invertible sheaves. (This is a case when inclusions of invertible sheaves does not mean what people normally mean by inclusion of line bundles, which are always isomorphisms.) Its cokernel is supported in dimension 0:

$$0 \to f^*\Omega_{C'} \to \Omega_C \to [\text{dimension } 0] \to 0.$$

The divisor R corresponding to those points (with multiplicity), is called the *ramification divisor*.

We can study this in local coordinates. We don't have the technology to describe this precisely yet, but you might still find this believable. If the map at $q \in C'$ looks like $u \mapsto u^n = t$, then $dt \mapsto d(u^n) = nu^{n-1}du$, so dt when pulled back vanishes to order $n-1$. Thus branching of this sort $u \mapsto u^n$ contributes $n-1$ to the ramification divisor. (More correctly, we should look at the map of Spec's of discrete valuation rings, and then u is a uniformizer for the stalk at q, and t is a uniformizer for the stalk at $f(q)$, and t is actually a unit times u^n. But the same argument works.)

Now in a recent exercise on pullbacks of invertible sheaves under maps of curves, we know that a degree of the pullback of an invertible sheaf is the degree of the map times the degree of the original invertible sheaf. Thus if d is the degree of the cover, $\deg \Omega_C =$

$d \deg \Omega_{C'} + \deg R$. Conclusion: if $C \to C'$ is a degree d cover of curves, then

$$\boxed{2g_c - 2 = d(2g_{C'} - 2) + \deg R}$$

Here are some applications.

Example. When I drew a sample branched cover of one complex curve by another, I showed a genus 2 curve covering a genus 3 curve. Show that this is impossible. (Hint: $\deg R \geq 0$.)

Example: Hyperelliptic curves. Hyperelliptic curves are curves that are double covers of $\mathbb{P}^1_k$. If they are genus g, then they are branched over $2g + 2$ points, as each ramification can happen to order only 1. (Caution: we are in characteristic 0!) You may already have heard about genus 1 complex curves double covering $\mathbb{P}^1$, branched over 4 points.

Application 1. First of all, the degree of R is even: any cover of a curve must be branched over an even number of points (counted with multiplicity).

Application 2. The only connected unbranched cover of $\mathbb{P}^1_k$ is the isomorphism. Reason: if $\deg R = 0$, then we have $2 - 2g_C = 2d$ with $d \geq 1$ and $g_c \geq 0$, from which $d = 1$ and $g_C = 0$.

Application 3: Luroth's theorem. Suppose $g(C) = 0$. Then from the Riemann-Hurwitz formula, $g(C') = 0$. (Otherwise, if $g_{C'}$ were at least 1, then the right side of the Riemann-Hurwitz formula would be non-negative, and thus couldn't be -2, which is the left side. This has a non-obvious algebraic consequence, by our identification of covers of curves with field extensions (class 28 Theorem 1.5). Hence all subfields of $k(x)$ containing k are of the form $k(y)$ where $y = f(x)$. (Here we have the hypothesis where k is algebraically closed. We'll patch that later.) Kirsten said that an algebraic proof was given in Math 210.

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