FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 30

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CONTENTS

Last day: More curves. Cohomology take 1.

Today: Cohomology continued. Hilbert functions and Hilbert polynomials.

1. LEFT-OVER: DEGREE OF A CARTIER DIVISOR ON A PROJECTIVE CURVE

As always, there is something small that I should have said last day. Suppose D is an effective Cartier divisor on a projective curve, or a Cartier divisor on a projective non-singular curve (over a field k). (I should really say: suppose D is a Cartier divisor on a projective curve, but I don't think I defined Cartier divisors in that generality.) Then define the *degree* of D (denoted $\deg D$) to be the degree of the corresponding invertible sheaf.

Exercise. If D is an effective Cartier divisor on a projective nonsingular curve, say $D = \sum n_i p_i$, prove that $\deg D = \sum n_i \deg p_i$, where $\deg p_i$ is the degree of the field extension of the residue field at p_i over k.

(This is also now in the class 29 notes, where it belongs.)

2. COHOMOLOGY CONTINUED

Last day, I gave you lots of facts that we wanted cohomology to satisfy. Suppose X is a separated and quasicompact R-scheme. In particular, X can be covered by a finite number of affine open sets, and the intersection of any two affine open sets is another affine open

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set. We are going to define $H^i(X, \mathcal{F})$ for any quasicoherent sheaf $\mathcal{F}$ on X, that satisfies the following properties.

- $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$
- H^i is a contravariant functor in X and a covariant functor in $\mathcal{F}$.
- $H^i(X, \oplus_j \mathcal{F}_j) = \oplus_j H^i(X, \mathcal{F}_j)$: cohomology commutes with arbitrary direct sums.
- long exact sequences
- $H^i(\operatorname{Spec} R, \mathcal{F}) = 0$.
- If $X \hookrightarrow Y$ is a closed immersion, and $\mathcal{F}$ is a quasicoherent sheaf on X, then $H^i(X, \mathcal{F}) = H^i(Y, f_* \mathcal{F})$.
- $H^i(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(r))$ is something nice (we described it in a statement last day that we will prove today)

Last day, we defined these cohomology groups given the additional data of an affine open cover $\mathcal{U}$; I used the notation $H^i_{\mathcal{U}}(X, \mathcal{F})$. We'll start today by showing that this is independent of $\mathcal{U}$.

2.1. *Theorem/Definition. — Recall that X is quasicompact and separated. $H^i_{\mathcal{U}}(U, \mathcal{F})$ is independent of the choice of (finite) cover $\{U_i\}$. More precisely,*

(*) *for all k, for any two covers $\{U_i\} \subset \{V_i\}$ of size at most k, the maps $H^i_{\{V_i\}}(X, \mathcal{F}) \to H^i_{\{U_i\}}(X, \mathcal{F})$ induced by the natural maps of complex (1) are isomorphisms.*

Define the Cech cohomology group $H^i(X, \mathcal{F})$ to be this group.

$$0 \to \bigoplus_{\substack{|I|=1 \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \cdots \to \bigoplus_{\substack{|I|=i \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \bigoplus_{\substack{|I|=i+1 \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \cdots. \tag{1}$$

I needn't have stated in terms of some k; I've stated it in this way so I can prove it by induction.

(For experts: we'll get natural quasiisomorphisms of Cech complexes for various $\mathcal{U}$.)

Proof. We prove this by induction on k. The base case is trivial. We need only prove the result for $\{U_i\}_{i=1}^n \subset \{U_i\}_{i=0}^n$, where the case $k = n$ is known. Consider the exact sequence of complexes

$$
\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \bigoplus_{\substack{|I|=i-1 \\ 0\in I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i \\ 0\in I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1 \\ 0\in I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \bigoplus_{\substack{|I|=i-1 \\ I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i \\ I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1 \\ I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \bigoplus_{\substack{|I|=i-1 \\ I\subset\{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i \\ I\subset\{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1 \\ I\subset\{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}
$$

We get a long exact sequence of cohomology from this. Thus by Exercise 5.2 of last day, we wish to show that the top row is exact. But the ith cohomology of the top row is precisely $H^i_{\{U_i\cap U_0\}_{i>0}}(U_i, \mathcal{F})$ except at step 0, where we get 0 (because the complex starts off $0 \to \mathcal{F}(U_0) \to \oplus_{j=1}^n \mathcal{F}(U_0 \cap U_j)$). So we just need to show that higher Cech groups of affine schemes are 0. Hence we are done by the following result. □

2.2. *Theorem. — The higher Cech cohomology* $H^i_{\mathcal{U}}(X, \mathcal{F})$ *of an affine* R*-scheme* X *vanishes (for any affine cover* $\mathcal{U}$*,* $i > 0$*, and quasicoherent* $\mathcal{F}$*).*

Serre describes this as a partition of unity argument.

A spectral sequence argument can make quick work of this, but I'd like to avoid introducing spectral sequences until I have to.

Proof. We want to show that the "extended" complex (where you tack on global sections to the front) has no cohomology, i.e. that

$$0 \to \mathcal{F}(X) \to \oplus_{|I|=1}\mathcal{F}(U_I) \to \oplus_{|I|=2}\mathcal{F}(U_I) \to \cdots \tag{2}$$

is exact. We do this with a trick.

Suppose first that some U_i (say U_0) is X. Then the complex can be described as the middle row of the following short exact sequence of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \oplus_{|I|=1,0\in I}\mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2,0\in I}\mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \oplus_{|I|=1}\mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2}\mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \oplus_{|I|=1,0\notin I}\mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2,0\notin I}\mathcal{F}(U_I) & \longrightarrow & \cdots
\end{array}$$

The top row is the same as the bottom row, slid over by 1. The corresponding long exact sequence of cohomology shows that the central row has vanishing cohomology. (Topological experts will recognize a *mapping cone* in the above construction.)

We next prove the general case by sleight of hand. Say $X = \operatorname{Spec} S$. We wish to show that the complex of R-modules (2) is exact. It is also a complex of S-modules, so we wish to show that the complex of S-modules (2) is exact. To show that it is exact, it suffices to show that for a cover of $\operatorname{Spec} S$ by distinguished opens $D(f_i)$ ($1 \leq i \leq s$) (i.e. $(f_1, \dots, f_s) = 1$ in S) the complex is exact. (Translation: exactness of a sequence of sheaves may be checked locally.) We choose a cover so that each $D(f_i)$ is contained in some $U_j = \operatorname{Spec} R_j$. Consider the complex localized at f_i. As

$$\Gamma(\operatorname{Spec} R, \mathcal{F})_f = \Gamma(\operatorname{Spec}(R_j)_f, \mathcal{F})$$

(as this is one of the definitions of a quasicoherent sheaf), as $U_j \cap D(f_i) = D(f_i)$, we are in the situation where one of the U_i's is X, so we are done. $\square$

2.3. *Exercise.* Suppose $V \subset U$ are open subsets of X. Show that we have restriction morphisms $H^i(U, \mathcal{F}) \to H^i(V, \mathcal{F})$ (if U and V are quasicompact, and U hence V is separated). Show that restrictions commute. Hence if X is a Noetherian space, $H^i(;\mathcal{F})$ this is a contravariant functor from the category $\operatorname{Top}(X)$ to abelian groups. (For experts: this means that it is a presheaf. But this is not a good way to think about it, as its sheafification is 0, as it vanishes on the affine base.) The same argument will show more generally that for any map $f : X \to Y$, there exist natural maps $H^i(X, \mathcal{F}) \to H^i(X, f^*\mathcal{F})$; I should have asked this instead.

2.4. *Exercise.* Show that if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on separated and quasicompact X then we have natural maps $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$. Hence $H^i(X, \cdot)$ is a covariant functor from quasicoherent sheaves on X to abelian groups (or even R-modules).

In particular, we get the following facts.

1. If $X \hookrightarrow Y$ is a closed subscheme then $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$, as promised at start of our discussion on cohomology.

2. Also, if X can be covered by n affine open sets, then $H^i(X, \mathcal{F}) = 0$ for all quasicoherent $\mathcal{F}$, and $i \geq n$. In particular, $H^i(\operatorname{Spec} R, \mathcal{F}) = 0$ for $i > 0$.

3. Cohomology behaves well for arbitrary direct sums of quasicoherent sheaves.

2.5. Dimensional vanishing for projective k-schemes.

2.6. *Theorem. — Suppose X is a projective k-scheme, and $\mathcal{F}$ is a quasicoherent sheaf on X. Then $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$.*

In other words, cohomology vanishes above the dimension of X. We will later show that this is true when X is a *quasiprojective* k-scheme.

Proof. Suppose $X \hookrightarrow \mathbb{P}^N$, and let $n = \dim X$. We show that X may be covered by n affine open sets. Long ago, we had an exercise saying that we could find n Cartier divisors on $\mathbb{P}^N$ such that their complements $U_0, \ldots, U_n$ covered X. (We did this as follows. Lemma: Suppose $Y \hookrightarrow \mathbb{P}^N$ is a projective scheme. Then Y is Noetherian, and hence has a finite number of components. We can find a hypersurface H containing none of their associated points. Then H contains no component of Y, the dimension of $H \cap Y$ is strictly smaller than Y, and if $\dim Y = 0$, then $H \cap Y = \emptyset$.) Then U_i is affine, so $U_i \cap X$ is affine, and thus we have covered X with n affine open sets. □

Remark. We actually *need* n affine open sets to cover X, but I don't see an easy way to prove it. One way of proving it is by showing that the complement of an affine set is always pure codimension 1.

3. Cohomology of line bundles on projective space

I'll now pay off that last IOU.

3.1. *Proposition. —*

- $H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ *is a free R-module of rank $\binom{n+m}{n}$ if $i = 0$ and $m \geq 0$, and 0 otherwise.*
- $H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ *is a free R-module of rank $\binom{-m-1}{-n-m-1}$ if $m \leq -n-1$, and 0 otherwise.*
- $H^i(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m)) = 0$ *if $0 < i < n$.*

It is more helpful to say the following imprecise statement: $H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ should be interpreted as the homogeneous degree m polynomials in $x_0, \ldots, x_n$ (with R-coefficients), and $H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ should be interpreted as the homogeneous degree m Laurent polynomials in $x_0, \ldots, x_n$, where in each monomial, each x_i appears with degree at most -1.

Proof. The H^0 statement was an (important) exercise last quarter.

Rather than consider $\mathcal{O}(m)$ for various m, we consider them all at once, by considering $\mathcal{F} = \oplus_m \mathcal{O}(m)$.

Of course we take the standard cover $U_0 = D(x_0), \ldots, U_n = D(x_n)$ of $\mathbb{P}^n_R$. Notice that if $I \subset \{1, \ldots, n\}$, then $\mathcal{F}(U_I)$ corresponds to the Laurent monomials where each x_i for $i \notin I$ appears with non-negative degree.

We consider the H^n statement. $H^n(\mathbb{P}^n_R, \mathcal{F})$ is the cokernel of the following surjection

$$\oplus_{i=0}^n \mathcal{F}(U_{\{1,\ldots,n\}-\{i\}}) \to \mathcal{F}_{U_{\{1,\ldots,n\}}}$$

i.e.

$$\oplus_{i=0}^n R[x_0, \ldots, x_n, x_0^{-1}, \ldots, \widehat{x_i^{-1}}, \ldots x_n^{-1}] \to R[x_0, \ldots, x_n, x_0^{-1}, \ldots, x_n^{-1}].$$

This cokernel is precisely as described.

We last consider the H^i statement ($0 < i < n$). We prove this by induction on n. The cases $n = 0$ and 1 are trivial. Consider the exact sequence of quasicoherent sheaves:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\times x_n} \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0$$

where $\mathcal{F}'$ is analogous sheaf on the hyperplane $x_n = 0$ (isomorphic to $\mathbb{P}^{n-1}_R$). (This exact sequence is just the direct sum over all m of the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_R}(m-1) \xrightarrow{\times x_n} \mathcal{O}_{\mathbb{P}^n_R}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}_R}(m) \longrightarrow 0\ ,$$

which in turn is obtained by twisting the closed subscheme exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_R}(m-1) \xrightarrow{\times x_n} \mathcal{O}_{\mathbb{P}^n_R}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}_R}(m) \longrightarrow 0$$

by $\mathcal{O}_{\mathbb{P}^n_R}(m)$.)

The long exact sequence in cohomology gives us:

$$0 \longrightarrow H^0(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^0(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow H^0(\mathbb{P}^{n-1}_R, \mathcal{F}') \quad .$$

$$\longrightarrow H^1(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^1(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow H^1(\mathbb{P}^{n-1}_R, \mathcal{F}')$$

$$\cdots \longrightarrow H^{n-1}(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^{n-1}(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow H^{n-1}(\mathbb{P}^{n-1}_R, \mathcal{F}')$$

$$\longrightarrow H^n(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^n(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow 0$$

We will now show that this gives an isomorphism

$$\boxed{\times x_n : H^i(\mathbb{P}^n_R, \mathcal{F}) \to H^i(\mathbb{P}^n_R, \mathcal{F})} \tag{3}$$

for $0 < i < n$. The inductive hypothesis gives us this except for $i = 1$ and $i = n-1$, where we have to pay a bit more attention. For the first, note that $H^0(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow H^0(\mathbb{P}^{n-1}_R, \mathcal{F}')$ is surjective: this map corresponds to taking the set of all polynomials in $x_0, \ldots, x_n$, and setting $x_n = 0$. The last is slightly more subtle: $H^{n-1}(\mathbb{P}^{n-1}_R, \mathcal{F}') \to H^n(\mathbb{P}^n_R, \mathcal{F})$ is injective, and corresponds to taking a Laurent polynomial in $x_0, \dots, x_{n-1}$ (where in each monomial, each x_i appears with degree at most -1) and multiplying by x_n^{-1}, which indeed describes the kernel of $H^n(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^n(\mathbb{P}^n_R, \mathcal{F})$. (This is a worthwhile calculation! See the exercise after the end of this proof.) We have thus established (3) above.

We will now show that the localization $H^i(\mathbb{P}^n_R, \mathcal{F})_{x_n} = 0$. (Here's what we mean by localization. Notice $H^i(\mathbb{P}^n_R, \mathcal{F})$ is naturally a module over $R[x_0, \dots, x_n]$ — we know how to multiply by elements of R, and by (3) we know how to multiply by x_i. Then we localize this at x_n to get an $R[x_0, \dots, x_n]_{x_n}$-module.) This means that each element $\alpha \in H^i(\mathbb{P}^n_R, \mathcal{F})$ is killed by some power of x_i. But by (3), this means that $\alpha = 0$, concluding the proof of the theorem.

Consider the Cech complex computing $H^i(\mathbb{P}^n_R, \mathcal{F})$. Localize it at x_n. Localization and cohomology commute (basically because localization commutes with operations of taking quotients, images, etc.), so the cohomology of the new complex is $H^i(\mathbb{P}^n_R, \mathcal{F})_{x_n}$. But this complex computes the cohomology of $\mathcal{F}_{x_n}$ on the affine scheme U_n, and the higher cohomology of *any* quasicoherent sheaf on an affine scheme vanishes (by Theorem 2.2 which we've just proved — in fact we used the same trick there), so $H^i(\mathbb{P}^n_R, \mathcal{F})_{x_n} = 0$ as desired. □

3.2. *Exercise.* Verify that $H^{n-1}(\mathbb{P}^{n-1}_R, \mathcal{F}') \to H^n(\mathbb{P}^n_R, \mathcal{F})$ is injective (likely by verifying that it is the map on Laurent monomials we claimed above).

4. Application of cohomology: Hilbert polynomials and Hilbert functions; degrees

We've already seen some powerful uses of this machinery, to prove things about spaces of global sections, and to prove Serre vanishing. We'll now see some classical constructions come out very quickly and cheaply.

In this section, we will work over a field k. Define $h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$.

Suppose $\mathcal{F}$ is a coherent sheaf on a projective k-scheme X. Define the *Euler characteristic*

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathcal{F}).$$

We will see repeatedly here and later that while Euler characteristics behave better than individual cohomology groups. As one sign, notice that for fixed n, and $m \geq 0$,

$$h^0(\mathbb{P}^n_k, \mathcal{O}(m)) = \binom{n+m}{m} = \frac{(m+1)(m+2)\cdots(m+n)}{n!}.$$

Notice that the expression on the right is a polynomial in m of degree n. (For later reference, I want to point out that the leading term is $m^n/n!$.) But it is not true that

$$h^0(\mathbb{P}^n_k, \mathcal{O}(m)) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}$$

for *all* m — it breaks down for $m \leq -n-1$. Still, you can check that

$$\chi(\mathbb{P}^n_k, \mathcal{O}(m)) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}.$$

So one lesson is this: if one cohomology group (usual the top or bottom) behaves well in a certain range, and then messes up, likely it is because (i) it is actually the Euler characteristic which is behaving well *always*, and (ii) the other cohomology groups vanish in that range.

In fact, we will see that it is often hard to calculate cohomology groups (even h^0), but it is often easier calculating Euler characteristics. So one important way of getting a hold of cohomology groups is by computing the Euler characteristics, and then showing that all the *other* cohomology groups vanish. Hence the ubiquity and importance of *vanishing theorems*. (A vanishing theorem usually states that a certain cohomology group vanishes under certain conditions.)

The following exercise already shows that Euler characteristic behaves well.

4.1. *Exercise.* Show that Euler characteristic is additive in exact sequences. In other words, if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on X, then $\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$. (Hint: consider the long exact sequence in cohomology.) More generally, if

$$0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$$

is an exact sequence of sheaves, show that

$$\sum_{i=1}^{n} (-1)^i \chi(X, \mathcal{F}_i) = 0.$$

4.2. *Exercise.* Prove the *Riemann-Roch theorem* for line bundles on a nonsingular projective curve C over k: suppose $\mathcal{L}$ is an invertible sheaf on C. Show that $\chi(\mathcal{L}) = \deg \mathcal{L} + \chi(C, \mathcal{O}_C)$. (Possible hint: Write $\mathcal{L}$ as the difference of two effective Cartier divisors, $\mathcal{L} \cong \mathcal{O}(Z - P)$ ("zeros" minus "poles"). Describe two exact sequences $0 \to \mathcal{O}_C(-P) \to \mathcal{O}_C \to \mathcal{O}_P \to 0$ and $0 \to \mathcal{L}(-Z) \to \mathcal{L} \to \mathcal{O}_Z \otimes \mathcal{L} \to 0$, where $\mathcal{L}(-Z) \cong \mathcal{O}_C(P)$.)

If $\mathcal{F}$ is a coherent sheaf on X, define the *Hilbert function of* $\mathcal{F}$:

$$h_{\mathcal{F}}(n) := h^0(X, \mathcal{F}(n)).$$

The *Hilbert function of* X is the Hilbert function of the structure sheaf. The ancients were aware that the Hilbert function is "eventually polynomial", i.e. for large enough n, it agrees with some polynomial, called the *Hilbert polynomial* (and denoted $p_{\mathcal{F}}(n)$ or $p_X(n)$). In modern language, we expect that this is because the Euler characteristic should be a polynomial, and that for $n \gg 0$, the higher cohomology vanishes. This is indeed the case, as we now verify.

I ended by stating the following, which we will prove next day.

4.3. *Claim. — For* $n \gg 0$, $h^0(X, \mathcal{F}(n))$ *is a polynomial of degree equal to the dimension of the support of* $\mathcal{F}$. *In particular,* $h^0(X, \mathcal{O}_X(n))$ *is "eventually polynomial" with degree* = $\dim X$.

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